Competition With Atomic Users

Kostas Bimpikis* Asuman Ozdaglar**

***Abstract*— We study the efficiency of oligopoly equilibria in a model where firms compete over prices. The motivating examples are the allocation of network flows in a communication network or of traffic in a transportation network. Contrary to most related papers, we study the case when the users are atomic, i.e., each user controls a non-negligible fraction of the total traffic. We show that competition among profit maximizing firms can reduce the overall efficiency of the system, measured as the difference between users' willingness to pay and delay costs. In particular, we characterize a tight bound of $1 - \frac{N}{6(N+1)}$ on worst case efficiency in pure strategy equilibria, where N is the number of atomic users.**

I. Introduction

We consider the problem of price competition in the presence of congestion costs assuming that the end users are atomic, i.e., each user controls a non-negligible fraction of the total traffic. In particular, we study the following environment: N self-interested users wish to route their flow using a network of I parallel links. Each of these links is owned by a profit maximizing firm. Firm i sets a per unit of flow price on link i. Finally, congestion is modeled with the use of convex non-decreasing latency functions on each link.

The motivation for our setting comes mainly from telecommunication and transportation networks and the key feature is the negative externality that users exert on others due to congestion. Early work on characterizing the effects of this negative externality include the seminal paper by Pigou, [12], as well as work in transportation and telecommunication networks ([3]). Recently, there has been considerable effort in quantifying the efficiency loss that results from the selfish behavior of the participating agents. In particular, there is a growing literature on providing tight bounds on the *"price of anarchy"*, defined as the worst case ratio of performance at equilibrium over the socially optimal performance ([13], [6], [11], [10]). Related to our work are the papers [14],[2] and [1], which study multi-stage games, where firms compete over prices and over prices and capacities respectively. Our current work extends the aforementioned models in the case when the users are atomic, i.e. control a non negligible amount of flow.

Our paper is also related to [7] and [9], which study routing games with atomic users. The difference of our work is that we are focusing on a two stage game, where profit maximizing firms compete over prices and self interested users route their traffic after observing the prices set in the first stage of the game. Note that users in our model are anticipating the effects of their routing decision on prices and are not simply price takers, as in [2] and [1].

This research was supported by the National Science Foundation CAREER grant DMI-0545910.

*Operations Research Center, Massachusetts Institute of Technology, kostasb@mit.edu

**Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, asuman@mit.edu

II. Model

We consider a network with I parallel links, each of which is owned by a different firm. Let x_i denote the total flow on link i. We assume that each link has a flow-dependent latency function $l_i(x_i)$, which measures the delay as a function of the total flow on link i. The firms set per unit of flow prices on the links they own. In particular, p_i denotes the price set by firm i on link i.

We are interested in the problem of routing d units of flow across the I links. Contrary to some recent papers, we assume that there exists a finite number N of self-interested users. In this paper we focus on the symmetric user model, where each user controls $\frac{d}{N}$ units of flow, i.e. all atomic users control the same amount of flow. Let $\mathcal{N} = \{1, \cdots, N\}$ denote the set of users and $\mathcal{I} = \{1, \cdots, I\}$ denote the set of links. We also assume that users obtain utility R per unit of flow they send across the network and that the users choose not to send any additional flow if the effective cost exceeds their utility R.

Next we define the notion of flow equilibrium in this setting

Definition 1 For a given price vector $p \geq 0$, a matrix $x^{FE} \in \Re_+^{I \times N}$ is a *Flow Equilibrium* (FE) if for every $k \in \{1, \cdots, N\}$

$$x^{k,FE} \in \arg \max_{x^k \geq 0, \sum x_i^k \leq d/N} \sum_{i=1}^{I} u_i x_i^k, \tag{1}$$

where $u_i = R - l_i(\sum_{j \neq k} x_i^{j,FE} + x_i^k) - p_i$. Also x_i^j denotes the flow that user j allocates to link i.

We denote the set of flow equilibria at a given price p by $FE(p)$.

Assumption 1 For each $i \in \mathcal{I}$, the latency function l_i is convex, nondecreasing, twice continuously differentiable and satisfies $l_i(0) = 0$, i.e., all latency is due to flow of traffic, there are no fixed latency costs.

The assumptions of differentiability and zero latency at zero flow are adopted because they simplify the analysis. Our results extend when these assumptions are relaxed, however the corresponding analysis is more involved.

From the optimality conditions for (1) we obtain the following lemma, which will be key in the subsequent analysis.

Lemma 1 Let Assumption 1 hold. Moreover let

$$\mathrm{P} = \min_m \{p_m + l_m(x_m^*) + x_m^{k,*} l_m^{'}(x_m^*)\}.$$

Then, a nonnegative matrix $x^* \in FE(p)$ if and only if the following hold for all players $k \in \{1, \cdots, N\}$,

$$\begin{aligned} &p_i + l_i(x_i^*) + x_i^{k,*} l_i^{'}(x_i^*) = \mathrm{P}, \quad \forall\ i \text{ such that } x_i^{k,*} > 0 \\ &p_i + l_i(x_i^*) + x_i^{k,*} l_i^{'}(x_i^*) \leq R, \qquad \forall i, k \\ &\sum_{i \in \mathcal{I}} x_i^{k,*} \leq d/N. \end{aligned}$$

If the last inequality is strict then

$$p_i + l_i(x_i^*) + x_i^{k,*} l_i^{'}(x_i^*) = R.$$

The existence and continuity properties of flow equilibria can be studied based on their equivalence with optimal solutions of a particular convex optimization problem. This is stated formally in the following proposition.

Proposition 1 Let Assumption 1 hold. For any price vector $p \geq 0$, the set of flow equilibria for the given price vector, $F(p)$, is nonempty. Moreover, the correspondence F is upper semicontinuous.

Proof: Consider the following optimization problem

$$\begin{aligned} \text{maximize}_{x \geq 0} \quad & \textstyle\sum_{k=1}^{N} \sum_{i=1}^{I} (R - p_i) x_i^k - \\ & \textstyle\int_0^{x_i^k + X_i} [l_i(z) + l_i^{'}(z)(z - X_i)] dz \\ \text{subject to} \quad & \textstyle\sum x_i^k \leq d/N \quad \forall\ k, \end{aligned}$$

where $X_i = \sum_{j \neq k} x_i^j$. The existence and continuity properties of flow equilibria can be established by noting that the first order optimality conditions of the above (convex) problem are precisely the FE optimality conditions. ∎

Note that flow equilibria are not necessarily unique unless the latency functions are strictly increasing.

We next define the social problem and the social optimum, which is the flow allocation that would be chosen by a planner that has full information and full control over the network and whose objective is to maximize the total efficiency of the system.

Definition 2 A flow vector x^S is a *social optimum* if it an optimal solution of the *social problem*

$$\begin{aligned} \text{maximize}_{x \geq 0} \quad & \textstyle\sum (R - l_i(x_i)) x_i \\ \text{subject to} \quad & \textstyle\sum x_i \leq d \end{aligned}$$

From assumption 1 we deduce that the social problem has a continuous objective function and a compact constraint set, thus a social optimum exists. Finally, we define the value of the objective function in the social problem,

$$S(x) = \sum_i (R - l_i(x_i)) x_i$$

as the *social surplus*, i.e. the difference between the users' willingness to pay and the total latency.

III. Oligopoly Equilibrium

As mentioned in the model description, we assume that there exist I service providers, each of which owns a single link. Service provider i charges a per unit of flow price p_i on the link i it owns. Given the vector of prices set by the competitors, the profit of service provider i is

$$\Pi_i(p_i, p_{-i}, x) = p_i \sum_{j=1}^{N} x_i^j,$$

for $x \in FE(p_i, p_{-i})$.

The objective of each service provider is to maximize profits. Note that the profits of service provider i depend on the prices set by the other providers, i.e. p_{-i}. We refer to the game among service providers as the *price competition game*. Naturally we can define an equilibrium of that game as follows

Definition 3 A vector $(p^{OE}, x^{OE}) \geq 0$ is a pure strategy *Oligopoly Equilibrium* (OE) if $x^{OE} \in FE(p^{OE})$ and for all $s \in S$

$$\Pi_s(p_s^{OE}, p_{-s}^{OE}, x^{OE}) \geq \Pi_s(p_s, p_{-s}^{OE}, x),$$
$$\forall\ p_s \geq 0 \text{ and } \forall\ x \in FE(p_s, p_{-s}^{OE}).$$

We refer to p^{OE} as the *OE price*. Note that the equilibrium concept we defined here is stronger than the usual equilibrium concept used for multistage games, the subgame perfect equilibrium. However, we can show that the two solution concepts are equivalent in our setting. Next we state an existence result for pure strategy oligopoly equilibria.

Proposition 2 Let Assumption 1 hold. Further assume that the latency functions are linear. Then the price competition game has a pure strategy Oligopoly Equilibrium.

The above existence result cannot be extended to general latency functions.

Here is an additional assumption that is useful for the price characterization at equilibrium we provide in proposition 3.

Assumption 2 The latency functions l_i satisfy $\sum_{i=1}^{I} l_i^{'}(x_i) > 0$, i.e. at any point at least one latency function is strictly increasing.

The following lemma states that if one of the firms makes positive profits then all the firms make positive profits.

Lemma 2 Let (p^{OE}, x^{OE}) be a pure strategy OE. Also, let Assumption 1 hold. Let Π_i denote the profit of service provider i at equilbrium. If $\Pi_i > 0$ for some $i \in \mathcal{I}$, then $p_j^{OE} x_j^{OE} > 0$ for all $j \in \mathcal{I}$.

Next we provide a characterization of equilibrium prices which is essential in our efficiency analysis.

Proposition 3 Let (p^{OE}, x^{OE}) be a pure strategy OE such that $p_i^{OE} x_i^{OE} > 0$ for some $i \in \mathcal{I}$. Let Assumptions 1 and 2 hold. We have

$$p_i^{OE} = \begin{cases} \frac{x_i^{OE}}{N} \quad [(N+1)l_i^{'} + x_i^{OE} l_i^{''}] \\ \qquad \text{if } l_j^{'} = l_j^{''} = 0 \text{ for some } j \neq i \\ \\ \min\{A, B\} \end{cases}$$

where

$$A = \begin{array}{l} \frac{x_i^{OE}}{N} \quad [(N+1)l_i^{'} + x_i^{OE} l_i^{''}] \\ \qquad + \frac{x_i^{OE}}{N} \left(\sum_{j\neq i} \frac{1}{(N+1)l_j^{'} + x_j^{OE} l_j^{''}} \right)^{-1} \end{array}$$

$$B = \min_k \{R - l_i - l_i^{'} x_i^{k,OE}\}$$

and x_i^{OE} denotes the total flow on link i at equilibrium. Note that for simplicity we omit the arguments in the expressions for the latency functions, i.e., we write l_i instead of $l_i(x_i^{OE})$.

Before moving on with the proof it is worth noting a few things.

- When $N \to \infty$ the price characterization above reduces to the one for nonatomic players in [2].
- Note that for this price characterization we used the fact that users have the same amount of flow. This allows us to assert that at equilibrium the effective costs for each player and link are equal, i.e.,

$$p_i + l_i + x_i^{k,OE} l_i^{'} = p_j + l_j + x_j^{k,OE} l_j^{'}, \forall\ i, j, k.$$

Proof: We will show the result for M service providers (links) and N atomic users, each controlling a non negligible amount of flow. Without loss of generality we will obtain an expression for the equilibrium price of link 1 (the same will be true for any i). Here is the optimization problem that firm 1 solves for choosing its price.

$$\begin{array}{ll} \max & p_1(x_1^1 + \cdots x_1^M) \\ \\ \text{s. t.} & \sum_{j=1}^{M} x_j^k \leq d^k \\ & \text{ for all atomic users } k \in \mathcal{N} \\ \\ & p_1 + l_1(x_1) + x_1^k l_1^{'}(x_1) = p_i + l_i(x_i) + x_i^k l_i^{'}(x_i) \\ & \text{ for all atomic users } k \in \mathcal{N} \text{ and all firms } i \neq 1 \end{array}$$

The analysis proceeds with examining the first order optimality conditions of the above optimization problem. In the following λ_{ij} corresponds to the langrange multiplier for the constraint that involves player i ($i \in \{1, \cdots, N\}$) and link j ($j \in \{2, \cdots, M\}$). In particular,

$$x_1 = \sum_{i,j} \lambda_{ij} \quad (2)$$

$$p_1 - \theta_k - \sum_{i=1}^{N} [\sum_{j=2}^{M} \lambda_{ij} (l_1^{'} + x_1^i l_1^{''} + \mathbf{1}_{k=j} l_1^{'})] = 0 \quad (3)$$

for all players k

(total N equations.)

$$\theta_k = \lambda_{kj}(2l_j^{'} + x_j^1 l_j^{''}) + \sum_{r\neq 1} [\lambda_{rj}(l_j^{'} + x_j^r l_j^{''})] \quad (4)$$

for all links $j \in \{2, \cdots, M\}$ and all players k

(total $N \cdot (M-1)$ equations.)

Under our assumptions the solution of the above system of equations yields the following.

$$p_i = \begin{array}{l} \frac{x_i^{OE}}{N} \quad [(N+1)l_i^{'} + x_i^{OE} l_i^{''}] + \\ \frac{x_i^{OE}}{N} \left(\sum_{j\neq i} \frac{1}{(N+1)l_j^{'} + x_j^{OE} l_j^{''}} \right)^{-1} \end{array},$$

or

$$p_i^{OE} = \frac{x_i^{OE}}{N} [(N+1)l_i^{'} + x_i^{OE} l_i^{''}].$$

when $l_j^{'} = l_j^{''} = 0$ for some $j \neq i$.

Finally we consider the case when the effective cost for some player k on link i is equal to the reservation utility R. Then,

$$p_i = R - l_i(x_i^{OE}) - l_i^{'}(x_i^{OE})x_i^{k,OE}.$$

Combining the preceding relations yields the desired price characterization. ■

IV. Efficiency Analysis

This section contains our main result, which provides tight bounds on the inefficiency of oligopoly equilibria. We consider only price competition games that have pure strategy equilibria. For such games, we define the efficiency metric as:

$$r_I(\{l_i\}, x^{OE}) = \frac{R - \sum_{i\in\mathcal{I}} x_i^{OE} \cdot l_i(x_i^{OE})}{R - \sum_{i\in\mathcal{I}} x_i^{S} \cdot l_i(x_i^{S})},$$

where x^S is a social optimum given the latency functions and R is the utility users obtain per unit of flow they send across the network. We are interested in providing a lower bound on the efficiency metric in the worst case scenario, both in terms of parameter values as well as equilibrium selection. In particular, following recent literature on the "price of anarchy" we are interested in providing lower bounds for the following quantity

$$\inf_{\{l_i\}\in L_I} \inf_{x^{OE}\in OE(\{l_i\})} r_I(\{l_i\}, x^{OE}).$$

In what follows we are using techniques developed in [2]. The idea can be summarized in the following. We are interested in minimizing the efficiency metric over the space of allowable latency functions. The problem is generally infinite-dimensional, however we are able to bound its value by a the optimal value of a finite dimensional problem using the relations between the flows at social optimum and equilibrium as well as assumptions on the latency functions (eg. convexity). Our main result is the following:

Theorem 1 Let Assumptions 1 and 2 hold. Consider a network with I links, where each link is owned by a different provider. Then

$$r \geq 1 - \frac{N}{6(N+1)}$$

where N denotes the number of self interested users. Moreover the bound is tight, i.e. there exists latency functions that attain the lower bound.

Proof Sketch. We begin by restricting the class of latency functions we need to consider. In particular, the next two lemmas provide conditions under which the price of anarchy is 1, i.e., there is no loss of efficiency at the corresponding oligopoly equilibria. The proofs of the lemmas are omitted due to space constraints (see our paper [5]).

Lemma 3 Let (p^{OE}, x^{OE}) be a pure strategy OE such that $p_i^{OE} x_i^{OE} = 0$ for all $i \in I$. Then x^{OE} is a social optimum.

Lemma 4 Assume that

$$R\sum_{i\in I} x_i^s = \sum_{i\in I} l_i(x_i^s) x_i^s,$$

for some social optimum x_s. Then every $x^{OE} \in OE(\{l_i\})$ is a social optimum, implying that the price of anarchy, i.e. $r_I(\{l_i\}, x^{OE}) = 1$.

From now and for the rest of the proof, we restrict attention to a two-link network with two atomic users. The proof readily extends to the case of I links and N atomic users.

Given $\{l_i\} \in \mathcal{L}_2$, i.e. the latency functions of the two links, let $x^{OE} \in OE(\{l_i\})$ and let x^s be a social optimum. We get that the following optimization problem is a lower bound on the price of anarchy:

$$\begin{aligned}
\text{minimize} \quad & \frac{R - l_1 y_1^{OE} - l_2 y_2^{OE}}{R - l_1^S y_1^S - l_2^S y_2^S} \\
\text{subject to} \quad & l_i^S \leq y_i^S (l_i^S)^{'} && (5)\\
& l_i \leq y_i^{OE} (l_i)^{'} && (6)\\
& l_2^S + y_2^S (l_2^S)^{'} = l_1^S + y_1^S (l_1^S)^{'} && (7)\\
& l_1^S + y_1^S (l_1^S)^{'} \leq R && (8)\\
& y_1^S + y_2^S \leq 1 && (9)\\
& l_1 + l_1^{'}(y_1^s - y_1^{OE}) \leq l_1^S && (10)\\
& +\{\text{Oligopoly Equilibrium Constraints}\} && (11)
\end{aligned}$$

The problem above can be viewed as a finite dimensional problem, which essentially captures the equilibrium and social optimum characteristics of the problem of minimizing the efficiency metric defined in the beginning of the section. This implies that instead of optimizing over entire functions, we can optimize over the possible function values at equilibrium and at social optimum. The constraints of the problem guarantee that the values over which we are optimizing satisfy the necessary conditions for social optimality and equilibrium. In particular, constraints 5 and 6 follow from the convexity assumptions on l_1, l_2, constraints 7 and 8 follow from the optimality conditions for the social optimum. Condition 10 follows by the convexity of function l_1. Finally, the last set of constraints are the necessary conditions for a pure strategy OE and are characterized by lemma 3.

It can be seen that at the optimal solution of the above problem, we have $l_1 = l_1^{'} = 0$, thereby reducing the problem to the following problem (for details, see refer to [5]):

$$\begin{aligned} \text{minimize} \quad & 1 - \frac{l_2 y_2^{OE}}{R} \\ \text{subject to} \quad & l_2 \le y_2^{OE}(l_2)^{'} \\ & y_1^{OE} + y_2^{OE} = 1 \\ \\ & p_1 = p_2 + l_2(y_2^{OE}) + y_{2A}^{OE} l_2^{'}(y_2) \\ & p_1 = p_2 + l_2(y_2^{OE}) + y_{2B}^{OE} l_2^{'}(y_2) \\ & p_1 \le R \end{aligned}$$

The optimal value of this problem is

$$(\bar{l_2}, \bar{l_2^{'}}, y_1^{OE}, y_2^{OE}, y_{2A}^{OE}, y_{2B}^{OE}) = (R/3, R, 2/3, 1/3, 1/6, 1/6)$$

and therefore it follows that the efficiency metric is lower bounded by $8/9$ (which is equal to $1 - \frac{N}{6(N+1)}$ for $N = 2$).

Next we give an example which matches the lower bound, thus implying that the bound is tight.

Example 1 Consider a two link network. Let the total flow be $d = 1$ and the reservation utility be R. Assume that the latency functions are given by

$$l_1(x) = 0 \quad \text{and} \quad l_2(x) = x$$

Moreover suppose that user A controls 1/2 of the flow, whereas user B controls another $1/2$. It can be verified that the vector $(p_1^{OE}, p_2^{OE}) = (1, 1/2)$ with $(x_1^{OE}, x_2^{OE}) = (2/3, 1/3)$ is a pure strategy OE. On the other hand, the social optimum is $(x_1^s, x_2^s) = (1, 0)$. The price of anarchy in this example is equal to 8/9.

Similar arguments can be used to show the result for the general case of N atomic players and M service providers. ■

V. Conclusions and Future Work

We have analyzed a model of price competition between profit maximizing firms when the end users can be atomic, i.e., each user controls a non-negligible fraction of the total traffic. We showed that in the worst case the system's efficiency loss is bounded from above by $\frac{N}{6(N+1)}$, where N is the number of users in the system, when the users are symmetric, i.e., they control the same amount of flow. An obvious extension of our work is to prove a tight bound on the efficiency loss, when users are asymmetric. We conjecture that the same bound is true for this case.

Our work raises a number of challenging questions. In particular, in many cases firms do not just name a price for their links, but rather, take the market power of the respective users into account, i.e. the amount of traffic they are controlling, and negotiate the price separately with each one of them. It would be interesting to incorporate this feature into our model, for example using some kind of a bargaining solution.

Another interesting issue is considering hybrid models, i.e. where a fraction of the total flow is controlled by infinitesimal users (as in most of previous literature) and the rest is controlled by a number of atomic users. A motivating example would be a transportation network, which is used by private cars (infinitesimal users) and a large truck company (atomic user).

References

[1] D. Acemoglu, K. Bimpikis, and A. Ozdaglar, *Price and capacity competition*, submitted for publication, 2006.

[2] D. Acemoglu and A. Ozdaglar, *Competition and efficiency in congested markets*, Mathematics of Operations Research (2007).

[3] M. Beckmann, C.B. Mcguire, and C.B. Winsten, *Studies in the economics of transportation*, Yale University Press, 1956.

[4] D.P. Bertsekas, A. Nedic, and A.E. Ozdaglar, *Convex analysis and optimization*, Athena Scientific, Cambridge, Massachusetts, 2003.

[5] K. Bimpikis and A. Ozdaglar, *Price competition with atomic users*, Mimeo, 2007.

[6] J.R. Correa, A.S. Schulz, and N.E. Stier-Moses, *Selfish routing in capacitated networks*, Mathematics of Operations Research **29** (2002), no. 4, 961–976.

[7] ———, *On the inefficiency of equilibria in congestion games*, Proceedings of the 11th Conference on Integer Programming and Combinatorial Optimization, vol. 3509, 2005, pp. 167–181.

[8] A. Hayrapetyan, E. Tardos, and T. Wexler, *A network pricing game for selfish traffic*, Proceedings of Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing, 2005.

[9] ———, *The effect of collusion in congestion games*, STOC '06: Proceedings of the thirty-eighth annual ACM symposium on Theory of computing (New York, NY, USA), ACM, 2006, pp. 89–98.

[10] R. Johari and J. Tsitsiklis, *Network resource allocation and a congestion game*, Mathematics of Operations Research **29** (2004), no. 3, 407–435.

[11] G. Perakis, *The price of anarchy when costs are non-separable and asymmetric*, Proceedings of the 10th Conference on Integer Programming and Combinatorial Optimization, vol. 3064, 2004, pp. 46–58.

[12] A.C. Pigou, *The economics of welfare*, Macmillan, London, U.K., 1920.

[13] T. Roughgarden and E. Tardos, *How bad is selfish routing?*, Journal of the ACM **49** (2002), no. 2, 236–259.

[14] G.Y. Weintraub, R. Johari, and B. Van Roy, *Investment and market structure in industries with congestion*, Preprint, 2006.