Near-Potential Games: Geometry and Dynamics

Ozan Candogan, Asuman Ozdaglar and Pablo A. Parrilo *

September 6, 2011

Abstract

Potential games are a special class of games for which many adaptive user dynamics converge to a Nash equilibrium. In this paper, we study properties of near-potential games, i.e., games that are close in terms of payoffs to potential games, and show that such games admit similar limiting dynamics. We first present a distance notion in the space of games and study the geometry of potential games and sets of games that are equivalent, with respect to various equivalence relations, to potential games. We discuss how given an arbitrary game, one can find a nearby game in these sets. We then study dynamics in near-potential games by focusing on continuous-time fictitious play dynamics. We characterize the limiting behavior of this dynamics in terms of the level sets of the potential function of a close potential game and approximate equilibria of the game. Exploiting structural properties of approximate equilibrium sets, we strengthen our result and show that for games that are sufficiently close to a potential game, the sequence of mixed strategies generated by this dynamics converges to a small neighborhood of equilibria whose size is a function of the distance from the set of potential games. We also consider continuous-time ϵ-fictitious play dynamics, a variant of fictitious play dynamics where players update their strategies only when the utility improvement is larger than some fixed level ϵ. When the game is sufficiently close to a potential game and ϵ is small, we establish convergence of this dynamics to a small neighborhood of equilibria.

*All authors are with the Laboratory of Information and Decision Systems, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Room 32-D608 Cambridge, MA 02139. emails: { `candogan, asuman, parrilo @mit.edu` }

1 Introduction

Potential games play an important role in game-theoretic analysis because of their appealing structural and dynamic properties. One property which is particularly relevant in the justification and implementation of equilibria is that many reasonable adaptive user dynamics converge to a Nash equilibrium in potential games (see [25, 24, 14, 31]). This motivates the question whether such convergence behavior extends to larger classes of games. A natural starting point is to consider games which are "close" to potential games.

In this paper, we formalize this notion and study *near-potential* games. We start by defining a distance notion on the space of games, and present a convex optimization formulation for finding the closest potential game to a given game in terms of it. We also consider best-response (in pure and mixed strategies) and Von Neumann-Morgenstern equivalence relations, and focus on sets of games that are equivalent (with respect to these equivalence relations) to potential games. Two games that are equivalent with respect to these equivalence relations generate the same trajectories of strategy profiles under update rules such as best response dynamics and fictitious play. Therefore, sets of games that are equivalent to potential games share the same limiting dynamic behavior as potential games. Identifying such sets enable us to extend the favorable dynamic properties of potential games to a larger set of games. We investigate how one can find the closest games that belong to these sets, and establish that this requires solving a nonconvex optimization problem. Set of games that are Von Neumann-Morgenstern equivalent to potential games coincide with the set of weighted potential games, a generalization of potential games. We study the geometry of this set and the set of ordinal potential games which is another generalization of potential games [25]. We show that these sets are nonconvex subsets of the space of games, and hence finding the closest weighted or ordinal potential game also requires solving a nonconvex optimization problem. We provide a convex optimization relaxation for finding a close weighted potential game to a given game. This relaxation allows for obtaining a weighted potential game that is (weakly) closer to the original game than the closest potential game. Moreover, if the original game is a weighted potential game our formulation identifies that the closest weighted potential game to this game is itself.

Our next set of results use this distance notion to provide a quantitative characterization of the limiting set of dynamics in games. We focus on continuous-time fictitious play, an update rule that is studied extensively in the literature as a reasonable model of adaptive user behavior (see [17, 14]). The trajectories of continuous-time fictitious play are characterized by differential equations, which involve each player updating its strategy according to a (singleton-valued) smoothened best response function. For potential games, the limiting behavior of the trajectories generated by this update rule can be analyzed using a Lyapunov function that consists of two terms: the potential function and a term related to the smoothing in the best responses. Previous work has established convergence of the trajectories to the equilibrium points of the differential equations [17]. We first show that these equilibrium points are contained in a set of approximate equilibria, which is a subset of a neighborhood of (mixed) Nash equilibria of the underlying game, when the smoothing factor is small (Theorem 4.1, and Corollary 4.1).

We then focus on near-potential games and show that in these games, the potential function of a nearby potential game increases along the trajectories of continuous-time fictitious play dynamics outside some approximate equilibrium set. This result is then used to establish convergence to a set of strategy profiles characterized by the approximate equilibrium set of the game, and the level sets of the potential function of a close potential game (Theorem 4.2).

Exploiting the properties of the approximate equilibrium sets we strengthen our result and show that if the original game is sufficiently close to a potential game (and the smoothing factor is small) then trajectories converge to a small neighborhood of equilibria whose size approaches zero as the distance from potential games (and the smoothing factor) goes to zero (Theorem 4.3). Our analysis relies on the following observation: In games with finitely many equilibria, for sufficiently small ϵ, the ϵ-equilibria are contained in small disjoint neighborhoods of the equilibria. Using this observation we first establish that after some time instant the trajectories of the continuous-time fictitious play dynamics visit only one such neighborhood. Then using the fact that the potential function increases outside a small approximate equilibrium set, and the Lipschitz continuity of the mixed extension of the potential function, we quantify how far the trajectories can get from this set.

We next define ϵ-fictitious play dynamics, a variant of continuous-time fictitious play dynamics where players update their strategies only if they have a utility[1] improvement opportunity of at least $\epsilon > 0$. This ϵ-stopping condition captures scenarios where agents (perhaps due to unmodelled costs) do not update their strategies unless there is a significant utility improvement possibility.[2] The differential equations that describe the evolution of trajectories for this update rule have discontinuous right hand sides. To deal with this issue, we adopt a solution concept that involves differential inclusions (at the points of discontinuity), and allow for multiple trajectories corresponding to a single initial condition. We establish that when the ϵ parameter is larger than the distance of the original game from a potential game (and the smoothing parameter), all trajectories of this update rule converge to an ϵ-equilibrium set (Theorem 5.2). Moreover, for small ϵ, this set is contained in a neighborhood of the equilibria of the game.

[1]In this paper, we use the terms utility and payoff interchangeably.

[2]It is documented in experimental economics that decision makers disproportionately stick with the status quo in settings that involve repeated decision making (see [27]). The update rule we study can be viewed as a model of status quo inertia.

Other than the papers cited above, our paper is related to a long line of literature studying convergence properties of various user dynamics in potential games: see [25, 31, 28] for better/best response dynamics, [24, 29, 20, 17, 29] for fictitious play and [6, 7, 1, 21] for logit response dynamics. It is most closely related to the recent papers [8, 10] and [9]. In [8, 10], we developed a framework for obtaining the closest potential game to a given game (using a distance notion slightly different than the one we employ in this paper). In [9], we studied convergence behavior of discrete time update processes in near-potential games. We showed that the trajectories of discrete time better/best response dynamics converge to approximate equilibrium sets while the empirical frequencies of fictitious play converge to a neighborhood of (mixed)Nash equilibrium. Moreover, the sizes of these sets diminish when the distance of the original game from the set of potential games goes to zero. This paper provides a more in depth study of the geometry of the set of potential games and sets of games that are equivalent to potential games, and investigates the limiting behavior of continuous-time update rules in near-potential games.

The rest of the paper is organized as follows: We present the game theoretic preliminaries for our work in Section 2. In Section 3, we show how to find a close potential game to a given game, and discuss the geometry of the sets of games that are equivalent to potential games. In Section 4, we analyze continuous-time fictitious play dynamics in near potential games. We introduce a variant of this update rule, and study its limiting behavior in Section 5. We close in Section 6 with concluding remarks and future work.

2 Preliminaries

In this section, we present the game-theoretic background that is relevant to our work. Additionally, we introduce some features of potential games and structural properties of mixed equilibria that are used in the rest of the paper.

2.1 Finite Strategic Form Games

Our focus in this paper is on finite strategic form games. A (noncooperative) finite game in strategic form consists of:

- A finite set of players, denoted by $\mathcal{M} = \{1, \dots, M\}$.
- Strategy spaces: A finite set of strategies (or actions) E^m, for every $m \in \mathcal{M}$.
- Utility functions: $u^m : \prod_{k \in \mathcal{M}} E^k \to \mathbb{R}$, for every $m \in \mathcal{M}$.

We denote a (strategic form) game by the tuple $\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$, the number of players in this game by $|\mathcal{M}| = M$, and the joint strategy space of this game by $E =$

$\prod_{m\in\mathcal{M}} E^m$. We refer to a collection of strategies of all players as a *strategy profile* and denote it by $\mathbf{p} = (p^1, \dots, p^M) \in E$. The collection of strategies of all players but the mth one is denoted by $\mathbf{p}^{-m}$.

The basic solution concept in a noncooperative game is that of a (pure) Nash Equilibrium (NE). A pure Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally, $\mathbf{p}$ is a Nash equilibrium if

$$u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m}) \leq 0,$$

for every $q^m \in E^m$ and $m \in \mathcal{M}$.

To address strategy profiles that are approximately a Nash equilibrium, we use the concept of ϵ-equilibrium. A strategy profile $\mathbf{p} \triangleq (p^1, \dots, p^M)$ is an ϵ-equilibrium ($\epsilon \geq 0$) if

$$u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m}) \leq \epsilon$$

for every $q^m \in E^m$ and $m \in \mathcal{M}$. Note that a Nash equilibrium is an ϵ-equilibrium with $\epsilon = 0$.

2.2 Potential Games

We next describe a particular class of games that is central in this paper, the class of potential games [25].

Definition 2.1. *Consider a noncooperative game $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$. If there exists a function $\phi : E \to \mathbb{R}$ such that for every $m \in \mathcal{M}$, $p^m, q^m \in E^m$, $\mathbf{p}^{-m} \in E^{-m}$,*

1. $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m})$, *then $\mathcal{G}$ is an* exact potential game.

2. $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = w_m(u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}))$, *for some strictly positive weight $w_m > 0$, then $\mathcal{G}$ is a* weighted potential game.

3. $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) > 0 \Leftrightarrow u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) > 0$, *then $\mathcal{G}$ is an* ordinal potential game.

The function ϕ is referred to as a potential function *of the game.*

This definition suggests that potential games are games in which the utility changes due to unilateral deviations for each player coincide with the corresponding change in the value of a global potential function ϕ. Note that every exact potential game is a weighted potential game with $w_m = 1$ for all $m \in \mathcal{M}$. From the definitions it also follows that every weighted potential game is an ordinal potential game. In other words, ordinal potential games generalize weighted potential games, and weighted potential games generalize exact potential games.

Definition 2.1 ensures that in exact, weighted and ordinal potential games unilateral deviations from a strategy profile that maximizes the potential function (weakly) decrease the utility of the deviating player. Thus, this strategy profile corresponds to a Nash equilibrium, and it follows that every ordinal potential game has a pure Nash equilibrium.

In this paper, our main focus is on exact potential games. The only exception is Section 3.3 where we discuss the geometries of weighted and ordinal potential games. For this reason, whenever there is no confusion, we refer to exact potential games as *potential games*.

We conclude this section by providing necessary and sufficient conditions for a game to be an exact or ordinal potential game. Before we formally state these conditions, we first provide some definitions, which will be used in Section 3.

Definition 2.2 (Path – Closed Path – Improvement Path)**.** *A* path *is a collection of strategy profiles $\gamma = (\mathbf{p}_0, \dots \mathbf{p}_N)$ such that $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ differ in the strategy of exactly one player. A path is a* closed path *(or a cycle) if $\mathbf{p}_0 = \mathbf{p}_N$. A path is an* improvement path *if $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$ where m_i is the player who modifies its strategy when the strategy profile is updated from $\mathbf{p}_{i-1}$ to $\mathbf{p}_i$.*

The transition from strategy profile $\mathbf{p}_{i-1}$ to $\mathbf{p}_i$ is referred to as step i of the path. We refer to a closed improvement path such that the inequality $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$ is strict for at least a single step of the path, as a *weak improvement cycle*. We say that a closed path is *simple* if no strategy profile other than the first and the last strategy profiles is repeated along the path. For any path $\gamma = (\mathbf{p}_0, \dots \mathbf{p}_N)$, let $I(\gamma)$ represent the "utility improvement" along the path, i.e.,

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(\mathbf{p}_i) - u^{m_i}(\mathbf{p}_{i-1}),$$

where m_i is the index of the player that modifies its strategy in the ith step of the path.

The following proposition provides an alternative characterization of exact and ordinal potential games. This characterization will be used when studying the geometry of sets of different classes of potential games (cf. Theorem 3.1).

Proposition 2.1 ([25], [30])**.** *(i) A finite game $\mathcal{G}$ is an exact potential game if and only if $I(\gamma) = 0$ for all simple closed paths γ.*

(ii) A finite game $\mathcal{G}$ is an ordinal potential game if and only if it does not include weak improvement cycles.

2.3 Mixed Strategies and ϵ-Equilibria

Our study of continuous-time fictitious play relies on the notion of mixed strategies and structural properties of mixed equilibrium sets in games. In this section we provide the relevant definitions and properties of mixed equilibria.

We start by introducing the concept of mixed strategies in games. For each player $m \in \mathcal{M}$, we denote by ΔE^m the set of probability distributions on E^m. For $x^m \in \Delta E^m$, $x^m(p^m)$ denotes the probability player m assigns to strategy $p^m \in E^m$. We refer to the distribution $x^m \in \Delta E^m$ as a *mixed strategy of player* $m \in \mathcal{M}$ and to the collection $\mathbf{x} = \{x^m\}_{m \in \mathcal{M}} \in \prod_m \Delta E^m$ as a *mixed strategy profile*. The mixed strategy profile of all players but the mth one is denoted by $\mathbf{x}^{-m}$. We use $||\cdot||$ to denote the standard 2-norm on $\prod_m \Delta E^m$, i.e., for $\mathbf{x} \in \prod_m \Delta E^m$, we have $||\mathbf{x}||^2 = \sum_{m \in \mathcal{M}} \sum_{p^m \in E^m} (x^m(p^m))^2$.

By slight (but standard) abuse of notation, we use the same notation for the mixed extension of utility function u^m of player $m \in \mathcal{M}$, i.e.,

$$u^m(\mathbf{x}) = \sum_{\mathbf{p} \in E} u^m(\mathbf{p}) \prod_{k \in \mathcal{M}} x^k(p^k), \tag{1}$$

for all $\mathbf{x} \in \prod_m \Delta E^m$. In addition, if player m uses some pure strategy q^m and other players use the mixed strategy profile $\mathbf{x}^{-m}$, the payoff of player m is denoted by

$$u^m(q^m, \mathbf{x}^{-m}) = \sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(q^m, \mathbf{p}^{-m}) \prod_{k \in \mathcal{M}, k \neq m} x^k(p^k).$$

Similarly, we denote the mixed extension of the potential function by $\phi(\mathbf{x})$, and we use the notation $\phi(q^m, \mathbf{x}^{-m})$ to denote the potential when player m uses some pure strategy q^m and other players use the mixed strategy profile $\mathbf{x}^{-m}$.

A mixed strategy profile $\mathbf{x} = \{x^m\}_{m \in \mathcal{M}} \in \prod_m \Delta E^m$ is a *mixed ϵ-equilibrium* if for all $m \in \mathcal{M}$ and $p^m \in E^m$,

$$u^m(p^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon. \tag{2}$$

Note that if the inequality holds for $\epsilon = 0$, then $\mathbf{x}$ is referred to as a *mixed Nash equilibrium* of the game. We use the notation $\mathcal{X}_\epsilon$ to denote the set of mixed ϵ-equilibria.

We conclude this section with two technical lemmas which summarize some continuity properties of the mixed equilibrium correspondence and the mixed extensions of potential and utility functions. These results appeared earlier in [9], and we refer the interested reader to this paper for the proofs.

Before we state these lemmas we first provide the relevant definitions (see [5, 16]).

Definition 2.3 (Upper Semicontinuous Function). *A function $g : X \to Y \subset \mathbb{R}$ is upper semicontinuous at x_*, if, for each $\epsilon > 0$ there exists a neighborhood U of x_*, such that $g(x) <$*

$g(x_*) + \epsilon$ for all $x \in U$. We say g is upper semicontinuous, if it is upper semicontinuous at every point in its domain.

Alternatively, g is upper semicontinuous if $\limsup_{x_n \to x_*} g(x_n) \leq g(x_*)$ for every x_* in its domain.

Definition 2.4 (Upper Semicontinuous Correspondence)**.** *A correspondence $g : X \rightrightarrows Y$ is upper semicontinuous at x_*, if for any open neighborhood V of $g(x_*)$ there exists a neighborhood U of x_* such that $g(x) \subset V$ for all $x \in U$. We say g is upper semicontinuous, if it is upper semicontinuous at every point in its domain and $g(x)$ is a compact set for each $x \in X$.*

Alternatively, when Y is compact, g is upper semicontinuous if its graph is closed, i.e., the set $\{(x, y) | x \in X, y \in g(x)\}$ is closed.

The first lemma establishes the Lipschitz continuity of the mixed extensions of the payoff functions and the potential function. The second lemma proves upper semicontinuity of the approximate equilibrium mapping.[3]

Lemma 2.1 ([9])**.** *Let $\nu : \prod_{m \in \mathcal{M}} E^m \to \mathbb{R}$ be a mapping from pure strategy profiles to real numbers. Its mixed extension is Lipschitz continuous with a Lipschitz constant of $M \sum_{p \in E} |\nu(\mathbf{p})|$ over the domain $\prod_{m \in \mathcal{M}} \Delta E^m$.*

Lemma 2.2 ([9])**.** *Let $g : \mathbb{R} \rightrightarrows \prod_{m \in \mathcal{M}} \Delta E^m$ be the correspondence such that $g(\alpha) = \mathcal{X}_\alpha$. This correspondence is upper semicontinuous.*

Upper semicontinuity of the approximate equilibrium mapping implies that for any given neighborhood of the ϵ-equilibrium set, there exists an $\epsilon' > \epsilon$ such that ϵ'-equilibrium set is contained in this neighborhood. In particular, this implies that every neighborhood of equilibria of the game contains an ϵ'-equilibrium set for some $\epsilon' > 0$. Hence, if there are finitely many equilibria, the disjoint neighborhoods of these equilibria contain the ϵ'-equilibrium set for a sufficiently small $\epsilon' > 0$. In Section 4, we use this observation to establish convergence of continuous-time fictitious play to small neighborhoods of equilibria of near-potential games.

3 Equivalence Classes and Geometry of Potential Games

In this section, we first introduce a distance notion in the space of games and show how to find the closest potential game to a given game with respect to this notion (Section 3.1).

[3]Here we fix the game, and discuss upper semicontinuity with respect to the ϵ parameter characterizing the ϵ-equilibrium set. We note that this is different than the common results in the literature which discuss upper semicontinuity of the equilibrium set with respect to changes in the utility functions of the underlying game (see [16]).

This notion is invariant under constant additions to payoffs of players, and the invariance can be used to define an equivalence relation in the space of games. We then introduce other equivalence relations for games, such as best response equivalence in pure and mixed strategies. These equivalence relations allow for identifying sets of games that have similar dynamical properties to potential games, and thereby extending our analysis to a larger set of nearby games. For a given game, we discuss how to find the closest games that are equivalent to potential games, and establish that the corresponding optimization formulations are nonconvex (Section 3.2). We establish a connection between the set of games that are equivalent to potential games and the set of weighted potential games and further study the geometry of sets of weighted and ordinal potential games. We show that these sets are nonconvex subsets of the space of games and provide a convex optimization relaxation for finding a close weighted potential game to a given game (Section 3.3).

3.1 Maximum Pairwise Difference and Near-Potential Games

We start this section by formally defining a notion of distance between games.

Definition 3.1 (Maximum Pairwise Difference)**.** *Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be two games with set of players $\mathcal{M}$, set of strategy profiles E, and collections of utility functions $\{u^m\}_{m\in\mathcal{M}}$ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ respectively. The maximum pairwise difference (MPD) between these games is defined as*

$$d(\mathcal{G},\hat{\mathcal{G}}) \triangleq \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \left|\left(u^m(q^m,\boldsymbol{p}^{-m}) - u^m(p^m,\boldsymbol{p}^{-m})\right) - \left(\hat{u}^m(q^m,\boldsymbol{p}^{-m}) - \hat{u}^m(p^m,\boldsymbol{p}^{-m})\right)\right|.$$

Note that the (pairwise) utility difference $u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})$ quantifies how much player m can improve its utility by unilaterally deviating from strategy profile $(p^m,\mathbf{p}^{-m})$ to strategy profile $(q^m,\mathbf{p}^{-m})$. Thus, the MPD captures how different two games are in terms of the utility improvements due to unilateral deviations. We refer to pairs of games with small MPD as *close games*, and games that have a small MPD to a potential game as *near-potential games*.

The MPD measures the closeness of games in terms of the difference of utility changes rather than the difference of their utility functions, i.e., using terms of the form

$$\left|\left(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\right) - \left(\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})\right)\right|$$

instead of terms of the form $|u^m(p^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})|$. This is because the difference in utility changes provides a better characterization of the strategic similarities (equilibrium and dynamic properties) between two games than the difference in utility functions. For instance consider two games with payoffs $u^m(\mathbf{p})$ and $u^m(\mathbf{p}) + \nu^m(\mathbf{p}^{-m})$ for every player m and strategy profile $\mathbf{p}$, where $\nu^m : \prod_{k\neq m} E^k \to \mathbb{R}$ is an arbitrary function. It can be seen from the definition of Nash equilibrium that despite a potentially nonzero difference of their utility functions, these two games share the same equilibrium set. Intuitively, since the ν^m term does not change the utility improvement due to unilateral deviations, it does not affect any of the strategic considerations in the game. While these two games have nonzero utility difference, the MPD between them is equal to zero.[4] Hence MPD identifies a strategic equivalence between these games.[5] We defer the discussion of other relevant notions of strategic equivalence to Section 3.2.

Using the distance notion defined above and Definition 2.1, we next formulate the problem of finding the closest potential game to a given game. Assume that a game with utility functions $\{u^m\}_m$ is given. The closest potential game (in terms of MPD) to this game, with payoff functions $\{\hat{u}^m\}_m$, and potential function ϕ can be obtained by solving the following optimization problem:

$$
\text{(P)}\qquad
\begin{aligned}
\min_{\phi,\{\hat{u}^m\}_m} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) \\
& \qquad\qquad - \big(\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})\big)\Big| \\
s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = \hat{u}^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \hat{u}^m(\bar{p}^m,\bar{\mathbf{p}}^{-m}), \\
& \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m.
\end{aligned}
$$

Note that the difference $(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})) - (\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m}))$ is linear in $\{\hat{u}^m\}_m$. Thus, the objective function is the maximum of such linear functions, and hence is convex in $\{\hat{u}^m\}_m$. The constraints of this optimization problem guarantee that the game with payoff functions $\{\hat{u}^m\}_m$ is a potential game with potential function ϕ. Note that these constrains are linear. Hence, the closest potential game (in terms of MPD) to a given game can be found by solving convex optimization problem (P).

3.2 Equivalence Classes of Games

In this section, we define different equivalence relations for games and focus on the sets of games that are equivalent to potential games. The games that belong to these sets have similar dynamic behavior to potential games under different update rules. This enables our analysis of dynamics in the subsequent sections to apply more broadly to games that will be nearby such sets.

We start by introducing an equivalence relation that involves MPD. The MPD between two games can be zero even when these games are not identical. For instance, as discussed

[4]Since two games that are not identical may have MPD equal to zero it follows that MPD is not a norm in the space of games. However, it can be easily shown that it is a seminorm.

[5]If two games have zero MPD, then the equilibrium sets of these games are identical. However, payoffs at equilibria may differ, and hence they may be different in terms of their efficiency (such as Pareto efficiency) properties (see [8]).

in the previous section the MPD between games with payoff functions $u^m(\mathbf{p})$ and $u^m(\mathbf{p}) + \nu^m(\mathbf{p}^{-m})$ equal to zero. We refer to games that have MPD equal to zero as *MPD-equivalent games*. It was established in [8] that two games with payoffs $\{u^m\}$ and $\{\hat{u}^m\}$ have identical (pairwise) utility differences (hence zero MPD) if and only if there exist functions $\nu^m : \prod_{k\neq m} E^k \to \mathbb{R}$ such that for all players m and strategy profiles $\mathbf{p}$, we have $u^m(\mathbf{p}) = \hat{u}^m(\mathbf{p}) + \nu^m(\mathbf{p}^{-m})$. This result provides a necessary and sufficient condition for MPD-equivalence between games.

We first consider games that are MPD-equivalent to potential games. It can be seen that any game that is MPD-equivalent to an (exact) potential game is also an (exact) potential game with the same potential function. Therefore, the set of games that are MPD-equivalent to potential games is the set of potential games itself.

We next introduce three additional equivalence relations that enable us to extend the set of potential games to larger sets with similar dynamic properties.

Definition 3.2. *Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be two games with set of players $\mathcal{M}$, set of strategy profiles E, and collections of utility functions $\{u^m\}_{m\in\mathcal{M}}$ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ respectively. These games are*

- Pure strategy best response equivalent *if for any player m and pure strategy profile $\boldsymbol{p}^{-m}$ we have* $\arg\max_{q^m\in E^m} u^m(q^m, \boldsymbol{p}^{-m}) = \arg\max_{q^m\in E^m} \hat{u}^m(q^m, \boldsymbol{p}^{-m})$,

- Mixed strategy best response equivalent *if for any player m and mixed strategy profile $\boldsymbol{x}^{-m}$ we have* $\arg\max_{q^m\in E^m} u^m(q^m, \boldsymbol{x}^{-m}) = \arg\max_{q^m\in E^m} \hat{u}^m(q^m, \boldsymbol{x}^{-m})$.

- VNM-equivalent *if for every player m, there exists a nonnegative constant $w^m > 0$, and a function $\nu^m : \prod_{k\neq m} E^k \to \mathbb{R}$ such that for any strategy profile $\boldsymbol{p}$, we have $u^m(\boldsymbol{p}) = w^m\hat{u}^m(\boldsymbol{p}) + \nu^m(\boldsymbol{p}^{-m})$.*

It can be easily seen from the definition that these are valid equivalence relations. The definition also suggests that VNM-equivalence reduces to MPD-equivalence when the weights $\{w^m\}$ are chosen to be equal to 1 for all players, i.e., two games that are MPD-equivalent are VNM-equivalent. Moreover, two games which are VNM-equivalent have identical best responses for every player m and every opponent mixed strategy profile $\mathbf{x}^{-m}$. This shows that games that are VNM-equivalent are mixed strategy best response equivalent. Clearly two games that are mixed strategy best response equivalent are pure strategy best response equivalent. Hence MPD, VNM, mixed and pure strategy best response equivalences define progressively more inclusive equivalence relations between games. An important consequence for our purposes is that these equivalence relations define a sequence of nested sets (by considering sets of games that are equivalent to potential games) containing the set of potential games. The paper [26] studies other relevant equivalence relations and it follows from their analysis that VNM-equivalence is strictly included in the mixed strategy best response equivalence, i.e., there are games that are mixed strategy best response equivalent, but not VNM-equivalent.[6]

Pure strategy best response equivalence is relevant in the context of best response dynamics (in pure strategies) since it leads to similar trajectories under this dynamics. Similarly, mixed strategy best response equivalence applies when considering dynamics that involve each player choosing a best response against their opponents' mixed strategies; an example of such dynamics is fictitious play (see [24], and also Section 4). Two games with payoff functions $\{u^m\}$ and $\{w^m u^m\}$, where $w^m > 0$ are per player weights, are VNM-equivalent (and hence mixed and pure strategy best response equivalent). Therefore, VNM-equivalence, while relevant for best-response type dynamics (in pure and mixed strategies), does not apply when considering dynamics that involve absolute values of payoff gains due to unilateral deviations (such as logit response, see [6, 7]). For such dynamics, it is natural to focus on classes of games which are MPD-equivalent.

We conclude this section by presenting optimization formulations for finding close games that are equivalent to potential games. As mentioned earlier, the set of games that are MPD-equivalent to potential games is the set of potential games itself. Thus, we focus on finding close games in the set of games which are pure/mixed strategy best response equivalent or VNM-equivalent to potential games.

Given a game with payoff functions $\{u^m\}$, we are interested in finding the closest game (in terms of MPD) which is pure/mixed strategy best response equivalent to a potential game. Since pure/mixed strategy best response equivalence is defined implicitly in terms of the solution of an optimization problem (note the $\arg\max$ operator in Definition 3.2), this

[6] The definition of best response equivalence in [26] is closely related to the equivalence relations defined here. In this paper, for each player m, the authors first define the function Λ_m such that

$$\Lambda_m(q_m, X|u^m) = \left\{ \lambda_m \in \Delta E^{-m} | \sum_{\mathbf{p}^{-m} \in E^{-m}} \lambda_m(\mathbf{p}^{-m})(u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m})) \geq 0 \text{ for all } p^m \in X \right\},$$

where ΔE^{-m} denotes the set of probability distributions over strategy profiles that belong to $\prod_{k \neq m} E^k$. That is for a given strategy $q^m \in E^m$, $\Lambda_m(q_m, X|u^m)$ denotes the set of distributions over $\prod_{k \neq m} E^k$ such that when the strategy profile of its opponents are drawn according to a distribution from this set, player m prefers strategy q^m to all other strategies in X. In [26], authors define two games to be best response equivalent if $\Lambda_m(q_m, E^m|u^m) = \Lambda_m(q_m, E^m|\hat{u}^m)$ for all players m and strategies $q^m \in E^m$. Note that this equivalence relation is contained in mixed strategy best response equivalence considered here, as it allows for distributions over $\prod_{k \neq m} E^k$ that are not necessarily mixed strategy profiles. In [26], it is established that there is nontrivial gap between best response equivalence and VNM-equivalence, i.e., there are games that are best response equivalent, but not VNM-equivalent. Since their best response equivalence is included in our mixed strategy best response equivalence, the same holds for our mixed strategy best response equivalence.

problem can be formulated as finding the closest such game with payoffs $\{v^m\}$ such that

$$\text{(P2)}\quad \begin{aligned} \min_{\phi,\{\hat{u}^m\}_m,\{v^m\}} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) \\ & \qquad\qquad - \big(v^m(q^m,\mathbf{p}^{-m}) - v^m(p^m,\mathbf{p}^{-m})\big)\Big| \\ s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = \hat{u}^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \hat{u}^m(\bar{p}^m,\bar{\mathbf{p}}^{-m}), \\ & \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m, \\ & \{v^m\}\ and\ \{\hat{u}^m\} \quad \text{are pure/mixed strategy best response equivalent.} \end{aligned}$$

On the other hand, games that are VNM-equivalent to potential games admit an explicit characterization in terms of payoffs. Assume that a game with payoff functions $\{u^m\}$ is given. Let $\{\hat{u}^m\}$ be a potential game with potential function ϕ, and assume that the closest game VNM-equivalent to this potential game has payoffs $\{v^m = w^m\hat{u}^m + \nu^m\}$, where for each m, ν^m is a function such that $\nu^m : \prod_{k\neq m} E^k \to \mathbb{R}$. The MPD between this game and the original game is given by

$$\max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) - \big(v^m(q^m,\mathbf{p}^{-m}) - v^m(p^m,\mathbf{p}^{-m})\big)\Big|.$$

Substituting $v^m = w^m\hat{u}^m + \nu^m$, and observing that ν^m terms cancel each other, the above equation takes the following form:

$$\max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) - w^m\left(\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})\right)\Big|.$$

Thus, the closest game to $\{u^m\}_m$ that is VNM-equivalent to a potential game, with payoff functions $\{\hat{u}^m\}_m$, and potential function ϕ can be obtained by solving the following optimization problem:

$$\text{(P3)}\quad \begin{aligned} \min_{\phi,\{w^m>0\}_m\{\hat{u}^m\}_m} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) \\ & \qquad\qquad - w^m\left(\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})\right)\Big| \\ s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = \hat{u}^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \hat{u}^m(\bar{p}^m,\bar{\mathbf{p}}^{-m}), \\ & \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m. \end{aligned}$$

The payoff functions in the equivalent game are given by $\{w^m\hat{u}^m + \nu^m\}$, where $\nu^m : \prod_{k\neq m} E^k \to \mathbb{R}$ is an arbitrary function.

Problem (P3) is a nonconvex optimization problem due to the nonlinear term $w^m\hat{u}^m$ in the objective function. This problem finds the closest game in the set of games that are VNM-equivalent to a potential game. It was shown in [26] that the set of games that are VNM-equivalent to potential games is the set of weighted potential games. To see this note from Definitions 2.1 and 3.2 that a game which is VNM-equivalent (with weights $\{w^m\}$) to an exact potential game with potential function ϕ is a weighted potential game with the same weights and potential function. Hence problem (P3) finds the closest weighted potential game to a given game. The next section studies the geometry of (exact, weighted, and ordinal) potential games to investigate the tractability of the problem of finding the closest weighted potential game.

3.3 Geometry of Potential Games

In this section, we discuss the geometry of sets of exact, weighted and ordinal potential games and discuss how to find a close weighted potential game to a given game. We first show that the sets of weighted and ordinal potential games are nonconvex, hence finding the closest (in the sense of MPD) potential game in these sets requires solving a nonconvex optimization problem. We then present a convex optimization relaxation for finding a close weighted potential game to a given game.

In this section, we denote the set of all games with set of players $\mathcal{M}$ and set of strategy profiles E by $\mathcal{G}_{\mathcal{M},E}$. Before we present our result on the sets of potential games, we introduce the notion of convexity for sets of games.

Definition 3.3. *Let $B \subset \mathcal{G}_{\mathcal{M},E}$. The set B is said to be convex if and only if for any two game instances $\mathcal{G}_1$, $\mathcal{G}_2 \in B$ with collections of utilities $u = \{u^m\}_{m\in\mathcal{M}}$, $v = \{v^m\}_{m\in\mathcal{M}}$ respectively*

$$\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{\alpha u^m + (1-\alpha)v^m\}_{m\in\mathcal{M}}\rangle \in B,$$

for all $\alpha \in [0,1]$.

Below we show that the set of exact potential games is a subspace of the space of games, whereas, the sets of weighted and ordinal potential games are nonconvex.

Theorem 3.1. *(i) The sets of exact potential games is a subspace of $\mathcal{G}_{\mathcal{M},E}$.*

(ii) The sets of weighted potential games, and ordinal potential games are nonconvex subsets of $\mathcal{G}_{\mathcal{M},E}$.

Proof. (i) Definition 2.1 implies that the set of exact potential games is the subset of space of games where the utility functions satisfy the condition

$$u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) = \phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}),$$

for some function ϕ, and all strategy profiles $\mathbf{p}$, and players m. Note that for each $\mathbf{p}$ and m this is a linear equality constraint on the utility functions $\{u^m\}$ and potential function ϕ. The set of all utility functions and potential functions that correspond to a potential game is the intersection of the sets defined by these linear equality constraints, and hence is a subspace. Projecting this set onto the collection of utility functions, or to $\mathcal{G}_{\mathcal{M},E}$, it follows that the set of exact potential games is also a subspace of the space of games.

(ii) We prove the claim by showing that the convex combination of two weighted potential games is not an ordinal potential game. This implies that the sets of both weighted and ordinal potential games are nonconvex since every weighted potential game is an ordinal potential game.

In Table 1 we present the payoffs and the potential function in a two player game, $\mathcal{G}_1$, where each player has two strategies. Given strategies of both players the first table shows payoffs of players (the first number denotes the payoff of the first player), the second table shows the corresponding potential function. In both tables the first column stands for actions of the first player and the top row stands for actions of the second player. Note that this game is a weighted potential game with weights $w_1 = 1$, $w_2 = 3$.

(u^1, u^2)	A	B
A	0,0	0,4
B	2,0	8,6

ϕ	A	B
A	0	12
B	2	20

Table 1: Payoffs and potential function in $\mathcal{G}_1$

Similarly, another game $\mathcal{G}_2$ is defined in Table 2. Note that this game is also a weighted potential game with weights $w_1 = 3$, $w_2 = 1$.

(u^1, u^2)	A	B
A	4,2	6,0
B	0,8	0,0

ϕ	A	B
A	20	18
B	8	0

Table 2: Payoffs and potential function in $\mathcal{G}_2$

In Table 3, we consider a game $\mathcal{G}_3$, in which the payoffs are averages (hence convex combinations) of payoffs of $\mathcal{G}_1$ and $\mathcal{G}_2$.

(u^1, u^2)	A	B
A	2,1	3,2
B	1,4	4,3

Table 3: Payoffs in $\mathcal{G}_3$

Note that this game has a weak improvement cycle:

$$(A, A) - (A, B) - (B, B) - (B, A) - (A, A).$$

From Proposition 2.1 (ii), it follows that $\mathcal{G}_3$ is not an ordinal potential game.

The above example shows that the sets of weighted and ordinal potential games with two players each of which has two strategies is nonconvex. For general n player games, the claim immediately follows by constructing two n player weighted potential games, and embedding $\mathcal{G}_1$ and $\mathcal{G}_2$ in these games. $\square$

The example provided in the proof of part (ii) of Theorem 3.1 also shows that the sets of games that are pure strategy or mixed strategy best response equivalent to potential games are nonconvex. To see this, note that $\mathcal{G}_1$ and $\mathcal{G}_2$ are weighted potential games thus mixed (and pure) strategy best response equivalent to exact potential games, but their convex combination, $\mathcal{G}_3$, cannot be pure or mixed strategy best response equivalent to an exact potential game, due to presence of an improvement cycle (note that all games that are pure/mixed strategy best response equivalent to $\mathcal{G}_3$ have the same improvement cycle, however Proposition 2.1 implies that potential games cannot have such cycles). Thus, finding the closest game that is mixed/pure strategy best response equivalent to an exact potential game involves solving a nonconvex optimization problem.

We next describe additional properties of weighted and ordinal potential games. It follows from Definition 2.1 that for any fixed set of weights, the set of weighted potential games is characterized by linear equalities similar to those used for exact potential games. Thus, it follows that the set of such games are also subspaces. Since the set of weighted potential games is union of such sets over all possible nonnegative weights, we conclude that the set of weighted potential games is an uncountable union of subspaces of $\mathcal{G}_{\mathcal{M},E}$. Given a game, the multiplication of the utilities by a positive scalar gives a game with the same weak improvement cycles. If the original game is an ordinal potential game and does not have a weak improvement cycle, the scaled game cannot have a weak improvement cycle. Thus, the scaled game is also an ordinal potential game. Therefore, we conclude that the set of ordinal potential games is a cone. By Theorem 3.1, it is a nonconvex cone.

We conclude this section by providing a convex optimization relaxation for finding a close weighted potential game to a given game. Since the set of weighted potential games coincides with the set of games that are VNM-equivalent to exact potential games, this relaxation can also be used to find a close game that is VNM-equivalent to an exact potential games.

The problem of finding the closest weighted potential game (with payoffs $\{\hat{u}^m\}$) to a given game (with payoffs $\{u^m\}$) can be formulated as follows:

$$\text{(W)}\quad \begin{aligned} \min_{\phi,\{w^m>0\}_m\{\hat{u}^m\}_m} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) \\ & \qquad\qquad - \big(\hat{u}^m(q^m,\mathbf{p}^{-m}) - \hat{u}^m(p^m,\mathbf{p}^{-m})\big)\Big| \\ s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = w^m\left(\hat{u}^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \hat{u}^m(\bar{p}^m,\bar{\mathbf{p}}^{-m})\right), \\ & \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m. \end{aligned}$$

Note that this optimization problem is an immediate reformulation of (P3), which can be used to find the closest game that is VNM-equivalent to a potential game. The nonconvexity in this problem stems from the equality constraints which involve the nonlinear terms $w^m \hat{u}^m$. Defining functions $v^m = w^m \hat{u}^m$, we obtain an equivalent problem:

$$
\text{(W2)} \quad
\begin{aligned}
\min_{\phi,\{w^m>0\}_m\{v^m\}_m} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(u^m(q^m,\mathbf{p}^{-m}) - u^m(p^m,\mathbf{p}^{-m})\big) \\
& \qquad\qquad - \frac{1}{w^m}\big(v^m(q^m,\mathbf{p}^{-m}) - v^m(p^m,\mathbf{p}^{-m})\big)\Big| \\
s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = v^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - v^m(\bar{p}^m,\bar{\mathbf{p}}^{-m}), \\
& \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m.
\end{aligned}
$$

This problem is still nonconvex due to the presence of $1/w^m$ term in the objective function. We present a convex optimization relaxation by scaling the objective function of the above problem by w^m for deviations corresponding to player m. This yields the following problem:

$$
\text{(W3)} \quad
\begin{aligned}
\min_{\phi,\{w^m\geq 1\}_m\{v^m\}_m} \quad & \max_{\mathbf{p}\in E, m\in\mathcal{M}, q^m\in E^m} \Big| \big(w^m u^m(q^m,\mathbf{p}^{-m}) - w^m u^m(p^m,\mathbf{p}^{-m})\big) \\
& \qquad\qquad - \big(v^m(q^m,\mathbf{p}^{-m}) - v^m(p^m,\mathbf{p}^{-m})\big)\Big| \\
s.t. \quad & \phi(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - \phi(\bar{p}^m,\bar{\mathbf{p}}^{-m}) = v^m(\bar{q}^m,\bar{\mathbf{p}}^{-m}) - v^m(\bar{p}^m,\bar{\mathbf{p}}^{-m}), \\
& \text{for all } m\in\mathcal{M},\ \bar{\mathbf{p}}\in E,\ \bar{q}^m\in E^m.
\end{aligned}
$$

Note that in the above formulation we require $w^m \geq 1$, unlike (W) and (W2) where we require $w^m > 0$. This is because, if we used the latter condition, then there would be feasible solutions of (W3) with objective values arbitrarily close to zero (by setting $v^m = \phi = 0$ and choosing weights arbitrarily close to zero) for any given game.

Since $(w^m u^m(q^m,\mathbf{p}^{-m}) - w^m u^m(p^m,\mathbf{p}^{-m})) - (v^m(q^m,\mathbf{p}^{-m}) - v^m(p^m,\mathbf{p}^{-m}))$ is linear in the problem variables ($\{w^m\}_m$ and $\{v^m\}_m$), the above is a convex optimization problem that can be used for finding a close weighted potential game to a given game. Given a solution of this problem, the game with utilities $\{v^m/w^m\}$ is a weighted potential game with weights $\{w^m\}$. The distance (in terms of MPD) of this game from the original game is bounded from above by the optimal value of (W3), since the objective function is a scaled version of MPD by weights (weakly) larger than 1.

Observe from Definition 2.1 that the weights and the potential function that correspond to a given weighted potential game are not unique: a weighted potential game with weights $\{w^m\}$ and potential function ϕ is also a weighted potential game with weights $\{\alpha w^m\}$ and potential function $\alpha\phi$, for any $\alpha > 0$. Thus, for any weighted potential game, it is possible to find a potential function corresponding to weights larger than 1. Observe that for a given weighted potential game, such weights and potential function constitute an optimal solution of (W3), and the corresponding objective value is equal to zero. Hence, our relaxation not only provides a tractable way of finding close weighted potential games, but it also determines whether a given game is a weighted potential game.

Consider an optimal solution of (W3), and observe that this is also a feasible solution of (W2). Thus, the optimal objective value of (W2) is weakly smaller than the value attained by this feasible solution. Hence the weighted potential game obtained by solving (W2) is closer to the original game than that obtained by solving (W3).[7] On the other hand, for weights equal to 1, (W3) is equivalent to (P). Thus, it follows that the optimal value of (W3) is smaller than or equal to that of (P), or equivalently the weighted potential game obtained by solving (W3) is (weakly) closer to the original game than the exact potential game obtained by solving (P). Moreover, for weighted potential games that are not exact potential games, as explained before (W3) has an optimal value of zero, whereas this is not the case for (P). Hence, in some cases (W3) leads to weighted potential games that are strictly closer (in terms of MPD) to the original game.

In the rest of the paper, we shift our attention to analysis of dynamics in near-potential games. In these sections, we do not discuss how a close potential game to a given game is obtained, but we just assume that a close exact potential game with potential function ϕ is known and the MPD between this game and the original game is δ. We provide characterization results on limiting dynamics for a given game in terms of ϕ and δ.

4 Continuous-time Fictitious Play Dynamics

In this section, we analyze convergence of continuous-time fictitious play dynamics in near-potential games. This update rule assumes that each agent continuously updates its strategy, by using its best response to its opponents' strategies. continuous-time fictitious play is closely related to its well-known discrete time counterpart through the stochastic approximation theory [3, 4]: under some conditions, "limit points" of discrete-time fictitious play are the "recurrent states" of continuous-time fictitious play. Hence, limiting behavior of discrete-time fictitious play can be characterized in terms of the limiting behavior of continuous-time fictitious play which in general admits a more tractable analysis than its discrete-time counterpart. Using this approach, convergence properties of different versions of discrete-time fictitious play have been established [4, 17, 14, 15].

We focus on a smoothened version of the continuous-time fictitious play dynamics previously studied in the literature [17, 14, 15]. It is known that for potential games this update rule converges [17, 29], and the convergence proof follows from a Lyapunov-function argument. Extending these results, we first establish that in potential games, provided that the

[7]Bounds on optimality loss due to our convex optimization relaxation, i.e., the optimality gap between the solutions of (W2) and (W3), are left for future work.

smoothing factor is small, the limiting point of fictitious play dynamics is contained in a small neighborhood of equilibria of the game. Then we show that the convergence properties of potential games extend to near-potential games.

Before we define continuous-time fictitious play dynamics, we introduce the notion of a smoothened best response. The smoothened (mixed strategy) best response of player m is a function $\tilde{\beta}^m : \prod_{k \neq m} \Delta E^k \to \Delta E^m$ such that

$$\tilde{\beta}^m(\mathbf{x}^{-m}) = \operatorname*{argmax}_{y^m \in \Delta E^m} u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m). \tag{3}$$

Here $H^m : \Delta E^m \to \mathbb{R}_+$ is a strictly concave function such that $\max_{y^m \in \Delta E^m} H^m(y^m) = \tau > 0$ for all $y^m \in \Delta E^m$. Since H^m is strictly concave and $u^m(y^m, \mathbf{x}^{-m})$ is linear in y^m, it follows that the smoothened best response is single valued. We refer to H^m as the smoothing function associated with the update rule, and τ as the smoothing parameter. A smoothing function that is commonly used in the literature is the entropy function [17, 14] , which is given by:

$$H^m(x^m) = -c \sum_{q^m \in E^m} x^m(q^m) \log(x^m(q^m)), \quad x^m \in \Delta E^m, \tag{4}$$

where $c > 0$ is a fixed constant. Note that for this function, if the value of c is small, then strategy $\tilde{\beta}^m$ is an approximate best response of player m.

We are now ready to provide a formal definition of continuous-time fictitious play.

Definition 4.1 (Continuous-Time Fictitious Play)**.** *Continuous-time fictitious play is the update rule where the mixed strategy of each player $m \in \mathcal{M}$ evolves according to the differential equation*

$$\dot{x}^m = \tilde{\beta}^m(\mathbf{x}^{-m}) - x^m. \tag{5}$$

Note that presence of the smoothing factor guarantees that the smoothened best response is single valued and continuous, and therefore the differential equation in (5) is well defined. We say that continuous-time fictitious-play converges to a set S, if starting from any mixed strategy profile, the trajectory defined by the above differential equation converges to S, i.e., $\inf_{\mathbf{x} \in S} ||\mathbf{x}_t - \mathbf{x}|| \to 0$ as $t \to \infty$.

If the original game is a potential game with potential function ϕ, $V(\mathbf{x}) = \phi(\mathbf{x}) + \sum_m H^m(x^m)$ is a strict Lyapunov function for the continuous-time fictitious play dynamics (see [17]). Thus, it follows that in potential games, fictitious play dynamics converges to the set of strategy profiles for which $\dot{x} = 0$, or equivalently $\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m = 0$ for all players m. Note that due to the presence of the smoothing term, this set does not coincide with the set of equilibria of the game. Our first result shows that if the smoothing term is bounded by τ, this set is contained in the τ-equilibrium set of the game.

Theorem 4.1. *In potential games, continuous-time fictitious play dynamics with smoothing parameter τ converges to $\mathcal{X}_\tau$, the set of τ-equilibria of the game.*

Proof. For each $\mathbf{x} \notin \mathcal{X}_\tau$, there exists a player m and strategy $y^m \in \Delta E^m$ such that $u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) > \tau$. Since $0 \leq H^m(z^m) \leq \tau$ for all $z^m \in \Delta E^m$ it follows that $u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m) - u^m(x^m, \mathbf{x}^{-m}) - H^m(x^m) > 0$, and thus $x^m \neq \tilde{\beta}^m(\mathbf{x}^{-m})$. Since fictitious play converges to the set of strategy profiles for which $\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m = 0$ for all players m, it follows that no $\mathbf{x} \notin \mathcal{X}_\tau$ belongs to the limiting set. Hence, the limiting points of fictitious play are contained in $\mathcal{X}_\tau$. □

The smoothing term is present in the definition of continuous-time fictitious play mainly for mathematical convenience (as it leads to a single valued, and continuous best response map). Thus, we can focus on analysis of convergence behavior when the smoothing factor is arbitrarily small. Using the structure of the mixed equilibrium sets of games and the above result, we next show that in potential games provided that the smoothing parameter is small, continuous time fictitious play dynamics converges to a small neighborhood of the equilibria of the game.

Corollary 4.1. *Consider a potential game $\mathcal{G}$, and let a constant $r > 0$ be given. There exists a sufficiently small $\tau > 0$ such that continuous-time fictitious play with a smoothing parameter τ, converges to r-neighborhood of the equilibria of the game.*

Proof. Lemma 2.2 (ii) implies that for small enough $\tau > 0$, $\mathcal{X}_\tau$ is contained in a r-neighborhood of the equilibria. Using this observation, the result immediately follows from Theorem 4.1. □

We next focus on near-potential games and investigate the convergence behavior of continuous-time fictitious play in such games. Our first result establishes that in near-potential games, starting from a strategy profile that is not an ϵ-equilibrium (where $\epsilon > M(\delta+\tau)$), the potential of a nearby potential game increases with rate at least $\epsilon - M(\delta+\tau)$. Using this result we also characterize the limiting behavior of continuous-time fictitious play in near-potential games in terms of the approximate equilibrium set of the game and the level sets of the potential function of a close potential game.

Theorem 4.2. *Consider a game $\mathcal{G}$ and let $\hat{\mathcal{G}}$ be a close potential game such that $d(\mathcal{G}, \hat{\mathcal{G}}) \leq \delta$. Denote the potential function of $\hat{\mathcal{G}}$ by ϕ, and the smoothing parameter by $\tau > 0$. Assume continuous-time fictitious play dynamics are employed in $\mathcal{G}$. Then,*

(i) $\dot{\phi}(\boldsymbol{x}_t) > \epsilon - M(\delta + \tau)$ if $\boldsymbol{x}_t \notin \mathcal{X}_\epsilon$

(ii) The trajectory of the update rule converges to the set of mixed strategy profiles which have potential larger than the minimum potential in the $M(\delta+\tau)$-equilibrium set of the game, i.e., $\mathbf{x}_t$ converges to $\{\mathbf{x} | \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_{M(\delta+\tau)}} \phi(\mathbf{y})\}$ as $t \to \infty$.

Proof. (i) When continuous-time fictitious play dynamics is used, players always modify their mixed strategies in the direction of their best responses. From the definition of mixed extension of ϕ it follows that

$$\dot{\phi}(\mathbf{x}_t) = \sum_m \nabla_m \phi^T(\mathbf{x}_t)(\tilde{\beta}^m(\mathbf{x}_t^{-m}) - x_t^m) = \sum_m (\phi(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m})), \tag{6}$$

where ∇_m denotes the collection of partial derivatives with respect to the strategies of player m, or equivalently the entries of the x^m vector.

Observe that if $\mathbf{x}_t \notin \mathcal{X}_\epsilon$, then there exists a player m and strategy y^m such that

$$u^m(y^m, \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) > \epsilon. \tag{7}$$

By definition of $\tilde{\beta}^m$ it follows that

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) + H^m(\tilde{\beta}^m(\mathbf{x}^{-m})) \geq u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m). \tag{8}$$

Since $0 \leq H^m(x^m) \leq \tau$, we obtain

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}) - \tau. \tag{9}$$

Equations (8) and (9) together with the definition of MPD and above inequality imply that

$$\begin{aligned} \phi(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m}) &\geq u^m(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) - \delta \\ &\geq u^m(y^m, \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) - \delta - \tau \\ &> \epsilon - \delta - \tau. \end{aligned} \tag{10}$$

Using the same equations, for players $k \neq m$, it follows that

$$\begin{aligned} \phi(\tilde{\beta}^k(\mathbf{x}_t^{-k}), \mathbf{x}_t^{-k}) - \phi(x_t^k, \mathbf{x}_t^{-k}) &\geq u^k(\tilde{\beta}^k(\mathbf{x}_t^{-k}), \mathbf{x}_t^{-k}) - u^k(x_t^k, \mathbf{x}_t^{-k}) - \delta \\ &\geq u^k(y^k, \mathbf{x}_t^{-k}) - u^k(x_t^k, \mathbf{x}_t^{-k}) - \delta - \tau \\ &\geq -\delta - \tau, \end{aligned} \tag{11}$$

where y^k denotes the best response of player k to $\mathbf{x}_t^{-k}$, i.e., $u^k(y^k, \mathbf{x}_t^{-k}) = \max_{z^k} u^k(z^k, \mathbf{x}_t^{-k})$. Thus, if $\mathbf{x}_t \notin \mathcal{X}_\epsilon$, then summing the above inequalities ((10) and (11)) over all players, and using (6) we obtain

$$\dot{\phi}(\mathbf{x}_t) > \epsilon - M(\delta + \tau) \tag{12}$$

(ii) Let $\epsilon > M(\delta + \tau)$ be given. The first part of the theorem implies that outside $\mathcal{X}_\epsilon$, the potential increases with a rate of at least $\epsilon - M(\delta + \tau) > 0$. Since the mixed extension of the potential function is a bounded function, it follows that starting from any strategy profile, the set $\mathcal{X}_\epsilon$ is reached in finite time. It is immediate that $\phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_\epsilon} \phi(\mathbf{y})$ for any $\mathbf{x} \in \mathcal{X}_\epsilon$. Since $\dot{\phi}(\mathbf{x}) > \epsilon - M(\delta + \tau) > 0$ for $\mathbf{x} \notin \mathcal{X}_\epsilon$, and $\dot{\phi}$ is bounded it follows that when the trajectory leaves $\mathcal{X}_\epsilon$, $\phi(\mathbf{x})$ cannot decrease below $\min_{\mathbf{y}\in\mathcal{X}_\epsilon}\phi(\mathbf{y})$. Thus, we conclude that after $\mathcal{X}_\epsilon$ is reached for the first time, the trajectory of dynamics satisfies $\phi(\mathbf{x}_t) \geq \max_{\mathbf{y}\in\mathcal{X}_\epsilon}\phi(\mathbf{y})$.

Therefore it follows that the trajectory converges to the set of mixed strategies $\{\mathbf{x}|\phi(\mathbf{x}) \geq \min_{\mathbf{y}\in\mathcal{X}_\epsilon}\phi(\mathbf{y})\}$. Since this is true for any $\epsilon > M(\delta+\tau)$, we obtain convergence to $\{\mathbf{x}|\phi(\mathbf{x}) \geq \min_{\mathbf{y}\in\mathcal{X}_{M(\delta+\tau)}}\phi(\mathbf{y})\}$. □

This theorem characterizes the limiting behavior of continuous-time fictitious play dynamics in near-potential games in terms of the approximate equilibrium set of the game, and the level sets of the potential function of a close potential game. As the deviation from a potential game increases, the set in which dynamics will be contained gradually becomes larger. Thus, characterization is more accurate for games that are closer to potential games.

For exact potential games, as we established in Corollary 4.1, the continuous-time fictitious play dynamics converges to a small neighborhood of the equilibria of the game, provided that the τ parameter is small. However, even for potential games (i.e., when $\delta = 0$) the above theorem provides a characterization of the limiting behavior only in terms of the level sets of the potential function, and hence cannot recover this convergence result for potential games.

We next strengthen the above convergence result by exploiting the properties of mixed approximate equilibrium sets in near-potential games. The feature of mixed equilibrium sets which plays a key role in our analysis was stated in Lemma 2.2. By considering the upper semicontinuity of the approximate equilibrium correspondence $g(\alpha)$ at $\alpha = 0$, this lemma implies that for small ϵ, the ϵ-equilibrium set is contained in a small neighborhood of equilibria.

It was established in part (i) of Theorem 4.2 that under continuous-time fictitious play dynamics the potential function of a nearby potential game (with MPD δ to the original game), evaluated at the current mixed strategy profile $\mathbf{x}$, increases when $\mathbf{x}$ is outside the $M(\delta+\tau)$-equilibrium set of the original game. As discussed above, if δ and τ are sufficiently small, then the $M(\delta+\tau)$-equilibria of the game will be contained in a small neighborhood of the equilibria. Thus, for sufficiently small δ and τ, it is possible to establish that the potential of a close potential game increases outside a small neighborhood of the equilibria of the game. In Theorem 4.3, we use this observation to show that for sufficiently small δ and τ the empirical frequencies of fictitious play dynamics converge to a neighborhood of an equilibrium. We state the theorem under the assumption that the original game has finitely many equilibria. This assumption generically holds, i.e., for any game a (nondegenerate) random perturbation of payoffs will lead to such a game with probability one (see [16]). When stating our result, we make use of the Lipschitz continuity of the mixed extension of the potential function, as established in Lemma 2.1.

Theorem 4.3. *Consider a game $\mathcal{G}$ and let $\hat{\mathcal{G}}$ be a close potential game such that $d(\mathcal{G},\hat{\mathcal{G}}) \leq \delta$. Denote the potential function of $\hat{\mathcal{G}}$ by ϕ, and the Lipschitz constant of the mixed extension of*

ϕ by L. Assume that $\mathcal{G}$ has finitely many equilibria, and in $\mathcal{G}$ players update their strategies according to continuous-time fictitious play dynamics.

There exists some $\bar{\delta}, \bar{\epsilon} > 0$, (which are functions of utilities of $\mathcal{G}$ but not δ) such that if $\delta + \tau < \bar{\delta}$, then the empirical frequencies of fictitious play converge to

$$\left\{ \mathbf{x} \,\middle|\, ||\mathbf{x} - \mathbf{x}_k^*|| \leq \frac{2MLf(M(\delta+\tau))}{\epsilon} + f(M(\delta+\tau)+\epsilon), \text{ for some equilibrium } \mathbf{x}_k^* \right\}, \quad (13)$$

for any $\bar{\epsilon} \geq \epsilon > 0$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semicontinuous function satisfying $f(x) \to 0$ as $x \to 0$.

The proof of this theorem can be found in the Appendix. As explained earlier, for small δ, τ and ϵ, the $M(\delta+\tau)+\epsilon$-equilibrium set of the game is contained in small neighborhoods of the equilibria of the game. If there are finitely many equilibria, these neighborhoods are disjoint and each of them contains a different component of the $M(\delta+\tau)+\epsilon$-equilibrium set. In the proof we show that the potential increases if the played strategy profile is outside this approximate equilibrium set. Then, we quantify the increase in the potential, when the trajectory leaves this approximate equilibrium set and returns back to it at a later time instant. Using this increase condition we show that after some time, the trajectory can visit the component of the approximate equilibrium set in the neighborhood of only a single equilibrium. This holds since, the increase condition guarantees that the potential increases significantly when the trajectory leaves the neighborhood of an equilibrium, and reaches to that of another equilibrium. Finally, using the increase condition one more time, we establish that if after time T, the trajectory visits the approximate equilibrium set only in the neighborhood of a single equilibrium, we can construct a neighborhood of this equilibrium, which contains the trajectory for all $t > T$. This neighborhood is expressed in (13).

If the original game is a potential game (hence $\delta = 0$), and the smoothing parameter τ is arbitrarily small, the above theorem implies (by choosing ϵ small as well, and using upper semicontinuity of f and the fact that $f(0) = 0$) that the trajectory of continuous-time fictitious play dynamics converges to a small neighborhood of the equilibria. Moreover, for any given $r > 0$, there exists a sufficiently small τ, such that ϵ can be chosen to guarantee $f(M(\delta+\tau)+\epsilon) \leq r/2$ and $2MLf(M(\delta+\tau))/\epsilon \leq r/2$. Thus, the above theorem recovers the convergence result of continuous-time fictitious play for potential games, stated in Corollary 4.1. Hence, we conclude that the convergence result of continuous-time fictitious play in near-potential games (Theorem 4.3) is a natural extension of the convergence result for potential games (Corollary 4.1).

5 Fictitious Play Dynamics with ϵ-stopping Condition

In this section, we introduce a variant of the continuous-time fictitious play dynamics, where players update their strategies only when there is possibility of significant utility improvement. This update rule is characterized with a strategy update threshold ϵ, i.e., only players who have at least ϵ utility improvement opportunity update their strategies. Presence of the ϵ threshold captures unmodeled decision making costs, which prevent players from updating their strategies, unless they can guarantee significant payoff improvement. We refer to this update rule as continuous-time ϵ-fictitious play.

Due to the presence of ϵ threshold for strategy updates, the differential equations describing this dynamical process has discontinuous right hand sides. Hence, in order to analyze this update rule, we introduce a proper solution concept that involves differential inclusions. We also present an invariance theorem which is used to characterize the limiting behavior of these differential inclusions. Using this machinery and exploiting properties of close potential games, we characterize the limiting behavior of continuous-time ϵ-fictitious play in near-potential games. In particular, we show that in near potential games trajectory of this update rule converges to a set of approximate equilibria. This set is contained in a small neighborhood of equilibria of the game provided that ϵ is small. Thus, we conclude that in near-potential games convergence results similar to those of continuous-time fictitious play can be obtained, even when a small ϵ threshold for strategy updates is present.

We start by providing a formal definition of continuous-time ϵ-fictitious play dynamics studied in this section. This definition involves the concept of smoothened best response, introduced in the previous section.

Definition 5.1 (Continuous-Time ϵ-Fictitious Play)**.** *Continuous-time ϵ-fictitious play is the update rule, where the mixed strategy of each player $m \in \mathcal{M}$ evolves according to the differential equation*

$$\dot{x}^m = \begin{cases} 0 & \text{if for all } y^m \in \Delta E^m,\ u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon, \\ \tilde{\beta}^m(\mathbf{x}^{-m}) - x^m & \text{otherwise.} \end{cases} \tag{14}$$

It can be seen from this definition that unlike fictitious play, in ϵ-fictitious play agents update their strategies only if it is possible to improve their utility by more than ϵ.

Note that the right hand side of (14) is discontinuous, and a proper solution concept should be adopted for its analysis. We introduce two relevant solution concepts that will be discussed in this section [13, 12, 11, 2].

Definition 5.2 (Krasovskii Solution - Filippov Solution)**.** *Given a differential equation $\dot{x} = f(x)$, with a discontinuous right hand side, (i) a Krasovskii solution is a solution of the* *differential inclusion:*

$$\dot{x} \in \bigcap_{\theta>0} \overline{co} f(x+\theta B), \tag{15}$$

(ii) a Filippov solution is a solution of the differential inclusion

$$\dot{x} \in \bigcap_{\theta>0} \bigcap_{\nu(\mathcal{N})=0} \overline{co} f(x+\theta B \setminus \mathcal{N}), \tag{16}$$

where, $x \in \mathbb{R}^n$, $\overline{co}$ S stands for the closure of the convex hull of set S, ν stands for the Lebesgue measure on $\mathbb{R}^n$ and B is the open unit ball in $\mathbb{R}^n$.

Intuitively, at the points of discontinuity, these definitions extend differential equations to differential inclusions, by allowing any right hand side which is in the convex hull of the discontinuous end points of the f function. Due to the intersection over sets of measure zero ($\nu(\mathcal{N}) = 0$) present in its definition, Filippov solutions disregard certain types of discontinuities[8], and lead to more "robust" solutions [18]. If f is a continuous function, both of these solutions reduce to regular differential equations.

Note that since in these solution concepts we have differential inclusions instead of differential equations, the trajectory corresponding to a given initial condition x_0 need not be unique. It can be seen from Definition 5.2 that the right hand side of the differential inclusion for Filippov solutions is a subset of that of Krasovskii solution, hence any trajectory obtained from a Filippov solution is also a Krasovskii solution. In the rest of the paper, we restrict our attention to Krasovskii solutions, and note that convergence for Filippov solutions immediately follows from convergence for Krasovskii solutions.

We next focus on the differential inclusion corresponding to the Krasovskii solutions of the ϵ-fictitious play dynamics, and characterize its properties. Note that the right hand side of (14) is discontinuous at $\mathbf{x} \in \prod_m \Delta E^m$, only when, for some player m and $z^m \in \Delta E^m$,

$$u^m(z^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) = \epsilon, \tag{17}$$

and for all other $y^m \in \Delta E^m$, $u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon$. For every player m, we define a function $g^m : \prod_m \Delta E^m \to 2^{[0,1]}$, such that

$$g^m(\mathbf{x}) = \begin{cases} \{0\} & \text{for all } y^m \in \Delta E^m,\ u^m(y^m, x^{-m}) - u^m(x^m, x^{-m}) < \epsilon \\ \{1\} & \text{for some } y^m \in \Delta E^m,\ u^m(y^m, x^{-m}) - u^m(x^m, x^{-m}) > \epsilon \\ [0,1] & \text{otherwise.} \end{cases} \tag{18}$$

Note that g^m is multivalued, if and only if (17) holds, or equivalently at the points of discontinuity of the right hand side of (14). Additionally, at these points, the definition of

[8]For instance, if one can obtain a continuous function f_c, changing the value of f function on a set of measure zero, then Filippov solution leads to a differential equation where the right hand side is f_c.

the Krasovskii solution implies that $\dot{x}^m$ belongs to the convex hull of $\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m$ and 0. Therefore, it follows that the differential inclusion F^m corresponding to the ϵ-fictitious play dynamics can be expressed as

$$\dot{x}^m \in F^m(\mathbf{x}) \qquad \text{for all } m \in \mathcal{M}, \tag{19}$$

where $F^m(\mathbf{x}) = \{\theta^m(\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m) | \theta^m \in g^m(\mathbf{x})\}$.

We say that continuous-time ϵ-fictitious play converges to a set S, if starting from any mixed strategy profile, *all* the trajectory defined by the above differential inclusion converges to S, i.e., $\inf_{\mathbf{x} \in S} ||\mathbf{x}_t - \mathbf{x}|| \to 0$ as $t \to \infty$.

We next present an invariance theorem, which will be used to establish convergence of ϵ-fictitious play dynamics in the above defined sense. Before we state the theorem, we introduce some additional notation. For any continuously differentiable function $V : \Omega \to \mathbb{R}$, we define the function $D_F V : \Omega \to \mathbb{R}$ associated with the multivalued function F, such that for all $x \in \Omega$

$$D_F V(x) = \sup_{\nu \in F(x)} \left\{ \lim_{z \to 0} \frac{1}{z} [V(x + z\nu) - V(x)] \right\}. \tag{20}$$

Note that $D_F V(x)$ is well defined as long as $x + z\nu \in \Omega$ for sufficiently small z and all $\nu \in F(x)$.

The following theorem generalizes La Salle's invariance theorem to differential inclusions (see [19, 22]) and will be used in our analysis of the continuous-time ϵ-fictitious play dynamics.

Theorem 5.1 ([22]). *Consider the differential inclusion, $\dot{x} \in F(x)$, where the trajectories of the dynamics are contained in a compact set and there exists a bounded set B such that $F(x) \subset B$ for all $x \in \Omega$. Assume that*

(i) There exists a continuously differentiable function $V : \Omega \to \mathbb{R}$ and a continuous function $W : \Omega \to \mathbb{R}$ satisfying

$$D_F V(x) \leq -W(x) \leq 0 \qquad \textit{for all } x \in \Omega. \tag{21}$$

(ii) V is bounded below.

Then, for any solution of the differential inclusion, trajectory x_t converges to $\{x \in \Omega | W(x) = 0\}$ as $t \to \infty$.

The collection of differential inclusions introduced in (19) can alternatively be written as

$$\dot{\mathbf{x}} \in F(\mathbf{x}), \tag{22}$$

where $F(\mathbf{x}) = \{F^m(\mathbf{x})\}_{m \in \mathcal{M}}$. Since $\tilde{\beta}^m(\mathbf{x}^{-m}), x^m \in \Delta E^m$, and $F^m(\mathbf{x}) = \{\theta^m(\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m) | \theta^m \in g^m(\mathbf{x})\}$, we have

$$\sum_{p^m \in E^m} \nu^m(p^m) = 0 \qquad \text{for all } \nu^m \in F^m(\mathbf{x}), \tag{23}$$

and hence $\sum_{p^m \in E^m} \dot{x}^m(p^m) = 0$. Additionally, since $\tilde{\beta}^m(\mathbf{x}^{-m}) \in \Delta E^m$ and $\dot{x}^m \in F^m(\mathbf{x})$, if at time t player m uses strategy p^m with probability 1, i.e., $x_t^m(p^m) = 1$, then $\dot{x}_t^m(p^m) \leq 0$, and similarly if it uses strategy p^m with probability 0, then $\dot{x}_t^m(p^m) \geq 0$. These facts imply that all trajectories generated by the differential inclusion in (22) are contained in the compact set $\prod_m \Delta E^m$, and $x^m + z\nu^m \in \Delta E^m$ for every $\nu^m \in F^m(\mathbf{x})$ and sufficiently small $z > 0$. Moreover, from the definition of $\tilde{\beta}^m(\mathbf{x}^{-m})$ it follows that $F(\mathbf{x})$ is a subset of the bounded set $\prod_m [-1,1]^{|E^m|}$. Thus, it can be seen that the differential inclusion in (22) satisfies conditions of Theorem 5.1 on properties of F, and trajectories of dynamics (conditions other than (i) and (ii)). We next show that conditions (i) and (ii) also hold for appropriately defined V and W functions, and use Theorem 5.1 to characterize the limiting behavior of continuous-time ϵ-fictitious play dynamics in near-potential games.

Theorem 5.2. *Consider a game $\mathcal{G}$ and let $\hat{\mathcal{G}}$ be a close potential game such that $d(\mathcal{G}, \hat{\mathcal{G}}) = \delta$. Denote the potential function of $\hat{\mathcal{G}}$ by ϕ, and the smoothing parameter by $\tau > 0$.*

If continuous-time ϵ-fictitious play dynamics are employed in $\mathcal{G}$, and $\epsilon > \delta + \tau$, then all trajectories of the update rule converge to the set of ϵ-equilibria of $\mathcal{G}$.

Proof. In order to prove the theorem, we define the functions $V : \prod_m \Delta E^m \to \mathbb{R}$ and $W : \prod_m \Delta E^m \to \mathbb{R}$ properly and use Theorem 5.1.

We define V such that $V(\mathbf{x}) = -\phi(\mathbf{x})$. Note that V is a bounded function that is also smooth and continuously differentiable. Additionally, for any $\mathbf{x} \in \prod_m \Delta E^m$, $m \in \mathcal{M}$ and $\nu^m \in F^m(\mathbf{x})$ we have $x^m + z\nu^m \in \Delta E^m$ for small enough z as explained before. Thus, the limit $\lim_{z \to 0} \frac{V(\mathbf{x}+z\nu) - V(\mathbf{x})}{z}$ exists for all $\mathbf{x} \in \prod_m \Delta E^m$ and $\nu \in F(\mathbf{x})$. Note that this quantity corresponds to the directional derivative of V in ν direction. Observing that $\prod_m \Delta E^m \subset \prod_m \mathbb{R}^{|E^m|}$, and writing the directional derivative of V in terms of the inner product of the relevant direction vector and the gradient of V, it follows that

$$\lim_{z \to 0} \frac{V(\mathbf{x} + z\nu) - V(\mathbf{x})}{z} = \nabla V^T(\mathbf{x})\nu = \sum_m \nabla_m V^T(\mathbf{x})\nu^m. \tag{24}$$

Here ν^m is the component of the ν vector corresponding to player m's strategies, $\nabla_m V$ denotes the vector of partial derivatives $\{\frac{\partial V}{\partial x^m(p^m)}\}_{p^m \in E^m}$, and $\frac{\partial V}{\partial x^m(p^m)}$ stands for the partial derivative of V with respect to the probability that player m uses strategy p^m. Therefore, using the characterization of F in (22) and (19), it follows that

$$\begin{aligned} D_F V(\mathbf{x}) &= \sup_{\nu \in F(\mathbf{x})} \left\{ \sum_m \nabla_m V^T(\mathbf{x})\nu^m \right\} \\ &= \sup_{\theta^k \in g^k(\mathbf{x}), \forall k} \sum_m \theta^m \nabla_m V^T(\mathbf{x})(\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m) \qquad (25) \\ &\leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} \theta^m \nabla_m V^T(\mathbf{x})(\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m). \end{aligned}$$

Note that $\frac{\partial V(\mathbf{x})}{\partial x^m(p^m)} = -\phi(p^m, \mathbf{x}^{-m})$, hence for any $y^m \in E^m$, it follows that $\nabla_m V^T(\mathbf{x}) y^m = -\phi(y^m, \mathbf{x}^{-m})$. Thus, the above inequality can be rewritten as:

$$D_F V(\mathbf{x}) \leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} -\theta^m(\phi(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - \phi(x^m, \mathbf{x}^{-m})). \tag{26}$$

The definition of g^m (see (18)) implies that θ^m can be chosen different than 0 only for agents for which

$$u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \geq \epsilon, \tag{27}$$

for some $y^m \in \Delta E^m$. Let m be such an agent, and y^m denote such a strategy. By definition of $\tilde{\beta}^m$ it follows that

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) + H^m(\tilde{\beta}^m(\mathbf{x}^{-m})) \geq u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m). \tag{28}$$

Since $0 \leq H^m(x^m) \leq \tau$, we obtain

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}) - \tau. \tag{29}$$

Subtracting $u^m(x^m, \mathbf{x}^{-m})$ from both sides, together with (27) the above inequality implies that

$$\begin{aligned} u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) &\geq u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) - \tau \\ &\geq \epsilon - \tau. \end{aligned} \tag{30}$$

Therefore, using the definition of MPD the above inequality implies that

$$\begin{aligned} \phi(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - \phi(x^m, \mathbf{x}^{-m}) &\geq u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) - \delta \\ &\geq \epsilon - \tau - \delta > 0. \end{aligned} \tag{31}$$

Since $g^k(\mathbf{x}) = \{0\}$, for agents which have $u^k(y^k, \mathbf{x}^{-k}) - u^k(x^k, \mathbf{x}^{-k}) < \epsilon$ for all y^k, and for all other agents the above inequality holds, (26) can be rewritten as

$$D_F V(\mathbf{x}) \leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} -\theta^m(\epsilon - \tau - \delta). \tag{32}$$

Let $\mathbf{x} \notin \mathcal{X}_\epsilon$, and k denote a player who can improve its payoff by strictly more than ϵ, unilaterally deviating from $\mathbf{x}$. Since $\theta^m \geq 0$ for $m \neq k$ and $g^k(\mathbf{x}) = \{1\}$ it follows that

$$D_F V(\mathbf{x}) \leq -(\epsilon - \tau - \delta) < 0. \tag{33}$$

Similarly, if $\mathbf{x} \in \mathcal{X}_\epsilon$, then (32) implies that

$$D_F V(\mathbf{x}) \leq 0. \tag{34}$$

Let $W : \prod_m \Delta E^m \to \mathbb{R}$ be a function such that $W(\mathbf{x}) = dist(\mathcal{X}_\epsilon, \mathbf{x})(\epsilon - \tau - \delta)$, where $dist(\mathcal{X}_\epsilon, \mathbf{x}) = \min_{\mathbf{y} \in \mathcal{X}_\epsilon} ||\mathbf{x} - \mathbf{y}||_\infty$ and $|| \cdot ||_\infty$ denotes the infinity norm in the Euclidean space

$\mathbb{R}^{\prod_m |E^m|}$, which contains $\prod_m \Delta E^m$. Since, $\mathcal{X}_\epsilon$ is a closed set and $\prod_m \Delta E^m \subset [0,1]^{\prod_m |E^m|}$ it follows that *dist* is a well defined continuous function, which is bounded by 1 for all $\mathbf{x} \in \prod_m \Delta E^m$. Thus, we have $W(\mathbf{x}) = 0$, for $\mathbf{x} \in \mathcal{X}_\epsilon$ and $0 < W(\mathbf{x}) \leq (\epsilon - \tau - \delta)$ for all $\mathbf{x} \in \prod_m \Delta E^m \setminus \mathcal{X}_\epsilon$. Hence, $W : \prod_m \Delta E^m \to \mathbb{R}$ is a continuous function, and as implied by (33) and (34), $D_F V(\mathbf{x}) \leq -W(\mathbf{x}) \leq 0$.

Since the trajectories of the dynamics are contained in the compact set $\prod_m \Delta E^m$ and $F(\mathbf{x})$ is a subset of the bounded set $\prod_m [-1,1]^{|E^m|}$, for all $\mathbf{x} \in \prod_m \Delta E^m$, Theorem 5.1, with the functions V and W introduced above, implies that the ϵ-fictitious play dynamics converge to the set for which $W(\mathbf{x}) = 0$. On the other hand, the definition of $W(\cdot)$ implies that this set is equivalent to $\mathcal{X}_\epsilon$, the set of ϵ-equilibria of $\mathcal{G}$, and the claim follows.

□

The above theorem implies that if the original game is close to a potential game, and the smoothing term is small (hence $\delta \approx 0$, and τ is small), ϵ can be chosen small to establish convergence of the continuous-time ϵ-fictitious play dynamics converges to a small approximate equilibrium set of the game. Moreover, as the deviation from a potential game increases, the set in which dynamics will be contained gradually becomes larger. By Lemma 2.2 for sufficiently small ϵ, the limiting approximate equilibrium set is contained in a small neighborhood of the equilibria of the game. Thus, we conclude that in near-potential games (and hence in potential games), even if players stop updating their strategies due to limited utility improvement opportunity, using a fictitious play type update rule leads to convergence to a small neighborhood of the equilibria for small ϵ, τ and δ.

6 Conclusions

In this paper, we study properties of near-potential games, and characterize the limiting behavior of continuous-time dynamics in these games. We first introduce a distance notion in the space of games, and study the geometry of sets of potential games and games that are equivalent to potential games. We also provide a framework for finding games that are close to potential games. Then we focus on continuous-time fictitious play dynamics, and in near-potential games characterize its limiting behavior in terms of the approximate equilibrium sets of the game, and the level sets of the potential function of a nearby potential game. The characterization is tighter when the original game is closer to a potential game. We strengthen our result by exploiting the structure of the mixed equilibrium sets, and showing that if the original game is sufficiently close to a potential game, then convergence to a small neighborhood of the equilibria can be established. We also study the convergence properties of ϵ-fictitious play, a variant of continuous-time fictitious play dynamics, where players update their strategies only if there is significant payoff improvement possibility. We show that in near-potential games, provided that the strategy update threshold ϵ and the distance between the game and a nearby potential game is sufficiently small, the trajectories of this update process converge to a small neighborhood of the equilibria of the game. Our results extend the known convergence properties of continuous-time fictitious play in potential games to near-potential games, and to settings where strategy updates take place only when there is sufficient utility improvement opportunity.

Our analysis and results motivate a number of interesting research questions. An interesting direction is to consider other learning dynamics, such as projection dynamics and replicator dynamics (see [28]), which are known to converge in potential games and investigate whether one can extend these convergence results to near-potential games. Other future work includes focusing on other classes of games with appealing dynamic properties, such as zero-sum games and supermodular games [23, 14, 31], and understanding through an analysis similar to the one in this paper, whether or not one can establish similar dynamic properties for nearby games.

References

[1] C. Alós-Ferrer and N. Netzer. The logit-response dynamics. *Games and Economic Behavior*, 68(2):413–427, 2010.

[2] J. Alvarez, I. Orlov, and L. Acho. An invariance principle for discontinuous dynamic systems with application to a Coulomb friction oscillator. *Journal of Dynamic Systems, Measurement, and Control*, 122:687, 2000.

[3] M. Benaim. A dynamical system approach to stochastic approximations. *SIAM Journal on Control and Optimization*, 34:437, 1996.

[4] M. Benaim and M.W. Hirsch. Mixed equilibria and dynamical systems arising from fictitious play in perturbed games. *Games and Economic Behavior*, 29(1-2):36–72, 1999.

[5] C. Berge. *Topological spaces.* Oliver & Boyd, 1963.

[6] L.E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387 – 424, 1993.

[7] L.E. Blume. Population games. In W.B. Arthur, S.N. Durlauf, and D.A. Lane, editors, *The economy as an evolving complex system II*, pages 425–460. Addison-Wesley, 1997.

[8] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo. Flows and decompositions of games: Harmonic and potential games. *Forthcoming in Mathematics of Operations Research*, 2011.

[9] O. Candogan, A. Ozdaglar, and P. A. Parrilo. Dynamics in Near-Potential Games. *ArXiv e-prints 1107.4386C*, July 2011.

[10] O. Candogan, A. Ozdaglar, and P.A. Parrilo. A projection framework for near-potential games. In *49th IEEE Conference on Decision and Control (CDC)*, pages 244–249, December 2010.

[11] F.H. Clarke, Y.S. Ledyaev, and R.J. Stern. Asymptotic Stability and Smooth Lyapunov Functions. *Journal of Differential Equations*, 149(1):69–114, 1998.

[12] K. Deimling. *Multivalued differential equations*. Walter de Gruyter, 1992.

[13] A.F. Filippov and F.M. Arscott. *Differential equations with discontinuous righthand sides*. Springer, 1988.

[14] D. Fudenberg and D.K. Levine. *The theory of learning in games*. MIT press, 1998.

[15] D. Fudenberg and S. Takahashi. Heterogeneous beliefs and local information in stochastic fictitious play. *Games and Economic Behavior*, 2008.

[16] D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, 1991.

[17] J. Hofbauer and W.H. Sandholm. On the global convergence of stochastic fictitious play. *Econometrica*, pages 2265–2294, 2002.

[18] T. Ito. A Filippov solution of a system of differential equations with discontinuous right-hand sides. *Economics Letters*, 4(4):349–354, 1979.

[19] H.K. Khalil. *Nonlinear systems*. Prentice Hall, 2002.

[20] J.R. Marden, G. Arslan, and J.S. Shamma. Joint strategy fictitious play with inertia for potential games. *IEEE Transactions on Automatic Control*, 54(2):208–220, 2009.

[21] J.R. Marden and J.S. Shamma. Revisiting log-linear learning: Asynchrony, completeness and a payoff-based implementation. *Games and Economic Behavior*, 2008.

[22] A.N. Michel, K. Wang, and B. Hu. *Qualitative theory of dynamical systems: the role of stability preserving mappings*. CRC, 2001.

[23] P. Milgrom and J. Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica: Journal of the Econometric Society*, 58(6):1255–1277, 1990.

[24] D. Monderer and L.S. Shapley. Fictitious play property for games with identical interests. *Journal of Economic Theory*, 68(1):258–265, 1996.

[25] D. Monderer and L.S. Shapley. Potential games. *Games and economic behavior*, 14(1):124–143, 1996.

[26] S. Morris and T. Ui. Best response equivalence. *Games and Economic Behavior*, 49(2):260–287, 2004.

[27] W. Samuelson and R. Zeckhauser. Status quo bias in decision making. *Journal of risk and uncertainty*, 1(1):7–59, 1988.

[28] W.H. Sandholm. *Population games and evolutionary dynamics.* MIT Press, Cambridge, MA, 2010.

[29] J.S. Shamma and G. Arslan. Unified convergence proofs of continuous-time fictitious play. *IEEE Transactions on Automatic Control*, 49(7):1137–1141, 2004.

[30] M. Voorneveld and H. Norde. A Characterization of Ordinal Potential Games. *Games and Economic Behavior*, 19(2):235–242, 1997.

[31] H.P. Young. *Strategic learning and its limits.* Oxford University Press, USA, 2004.

A Proofs

Proof of Theorem 4.3: Assume that $\mathcal{G}$ has $l > 0$ equilibria, denoted by $\mathbf{x}_1^*, \ldots, \mathbf{x}_l^*$. Define the minimum pairwise distance between the equilibria as $d \triangleq \min_{i \neq j} ||\mathbf{x}_i^* - \mathbf{x}_j^*||$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that

$$f(\alpha) = \max_{\mathbf{x} \in \mathcal{X}_\alpha} \min_{k \in \{1,\ldots,l\}} ||\mathbf{x} - \mathbf{x}_k^*||, \tag{35}$$

for all $\alpha \in \mathbb{R}_+$. Note that $\min_{k \in \{1,\ldots,l\}} ||\mathbf{x} - \mathbf{x}_k^*||$ is continuous in $\mathbf{x}$, since it is minimum of finitely many continuous functions. Moreover, $\mathcal{X}_\alpha$ is a compact set, since ϵ-equilibria are defined by finitely many inequality constraints of the form (2). Therefore, in (35) maximum is achieved and f is well-defined for all $\alpha \geq 0$. From the definition of f, it follows that the union of closed balls of radius $f(\alpha)$, centered at equilibria, contain α-equilibrium set of the game. Thus, intuitively, $f(\alpha)$ captures the size of a closed neighborhood of equilibria, which contains α-equilibria of the underlying game.

Let $a > 0$ be such that $f(a) < d/4$, i.e., every a-equilibrium is at most $d/4$ distant from an equilibrium of a game. Lemma 2.2 implies (using upper semicontinuity at $\alpha = 0$) that such a exists. Since d is defined as the minimum pairwise distance between the equilibria, it follows that a-equilibria of the game are contained in disjoint $f(a) < d/4$ neighborhoods around equilibria of the game, i.e., if $\mathbf{x} \in \mathcal{X}_a$, then $||\mathbf{x} - \mathbf{x}_k^*|| \leq f(a)$ for exactly one equilibrium $\mathbf{x}_k^*$. Moreover, for $a_1 \leq a$, since $\mathcal{X}_{a_1} \subset \mathcal{X}_a$, it follows that a_1-equilibria of the game are also contained in disjoint neighborhoods of equilibria.

We prove the theorem in 4 steps summarized below. First two steps explore the properties of function f, and define $\bar{\delta}$ and $\bar{\epsilon}$ present in the theorem statement. Last two steps are the main steps of the proof, where we establish convergence of fictitious play to a neighborhood of equilibria.

- *Step 1:* We first show that f is (i) weakly increasing, (ii) upper semicontinuous, and it satisfies (iii) $f(0) = 0$, (iv) $f(x) \to 0$ as $x \to 0$.

- *Step 2:* We show that there exists some $\bar{\delta} > 0$, $\bar{\epsilon} > 0$ such that the following inequalities hold:

$$M\bar{\delta} + \bar{\epsilon} < a, \tag{36}$$

and

$$f(M\bar{\delta} + \bar{\epsilon}) < \frac{d(a - M\bar{\delta})}{16ML}. \tag{37}$$

We will prove the statement of the theorem assuming that $0 \leq \delta + \tau < \bar{\delta}$ and establish convergence to the set in (13), for any $\bar{\epsilon} \geq \epsilon > 0$. As can be seen from the definition of a and f (see (35)), the first inequality guarantees that $\bar{\epsilon} + M\bar{\delta}$-equilibrium set is contained in disjoint neighborhoods of equilibria, and the second one guarantees that these neighborhoods are small.

- *Step 3:* In this step we prove that after some time fictitious play can visit the $\bar{\epsilon}+M(\delta+\tau)$-equilibrium set contained in the neighborhood of only one equilibrium.
- *Step 4:* In this step, using the fact that the neighborhood of only one equilibrium is visited after some time, we show that the trajectory converges to the set given in the theorem statement.

Next we prove each of these steps.

Step 1: By definition $\mathcal{X}_{\alpha_1} \subset \mathcal{X}_\alpha$ for any $\alpha_1 \leq \alpha$. Since the feasible set of the maximization problem in (35) is given by $\mathcal{X}_\alpha$, this implies that $f(\alpha_1) \leq f(\alpha)$, i.e., f is a weakly increasing function of its argument. Note that the feasible set of the maximization problem in (35) can be given by the correspondence $g(\alpha) = \mathcal{X}_\alpha$, which is upper semi continuous in α as shown in Lemma 2.2. Since as a function of $\mathbf{x}$, $\min_{k\in\{1,\dots,l\}} ||\mathbf{x}-\mathbf{x}_k^*||$ is continuous it follows from Berge's maximum theorem (see [5]) that for $\alpha \geq 0$, $f(\alpha)$ is an upper semicontinuous function.

The set $\mathcal{X}_0$ corresponds to the set of equilibria of the game, hence $\mathcal{X}_0 = \{\mathbf{x}_1^*, \dots, \mathbf{x}_l^*\}$. Thus, the definition of f implies that $f(0) = 0$. Moreover, upper semicontinuity of f implies that for any $\epsilon > 0$, there exists some neighborhood V of 0, such that $f(x) \leq \epsilon$ for all $x \in V$. Since, $f(x) \geq 0$ by definition, this implies that $\lim_{x\to 0} f(x)$ exists and equals to 0.

Step 2: Let $\bar{\delta}, \bar{\epsilon} > 0$ be small enough such that $M\bar{\delta} + \bar{\epsilon} < a/2$. Since $\lim_{x\to 0} f(x) = 0$, it follows that for sufficiently small $\bar{\delta}, \bar{\epsilon}$ we obtain $f(M\bar{\delta}+\bar{\epsilon}) < \frac{ad}{32ML} < \frac{(a-M\bar{\delta})d}{16ML}$.

Step 3: Assume that the trajectory of the dynamics leaves the component of the $\bar{\epsilon} + M(\delta+\tau)$-equilibrium set contained in the neighborhood of equilibrium $\mathbf{x}_k^*$, and enters the component of this approximate equilibrium set in the neighborhood of equilibrium $\mathbf{x}_l^*$. Since $\bar{\epsilon}+M(\delta+\tau) \leq \bar{\epsilon}+M\bar{\delta} < a$ in such an evolution the trajectory needs to leave the $f(a) < d/4$ neighborhood of equilibrium $\mathbf{x}_k^*$, and enter the $f(a)$ neighborhood of equilibrium $\mathbf{x}_l^*$.

Let t_1 denote the time instant the trajectory leaves the component of the $\bar{\epsilon}+M(\delta+\tau)$-equilibrium set around equilibrium $\mathbf{x}_k^*$, t_2 denote the instant it leaves the $f(a)$ neighborhood of $\mathbf{x}_k^*$, t_3 denote the instant it enters the $f(a)$ neighborhood of $\mathbf{x}_l^*$, and finally t_4 denote the instant it enters the component of the $\bar{\epsilon}+M(\delta+\tau)$-equilibrium set around equilibrium $\mathbf{x}_l^*$.

It follows from part (i) of Theorem 4.2 that

$$\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}), \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3}) \geq 0. \tag{38}$$

Observe that $\mathbf{x}_t$ is outside the $f(a)$ neighborhood of equilibria between t_2 and t_3, and this neighborhood contains $\mathcal{X}_a$. Since equilibria are at least d apart, and $f(a) < d/4$ we obtain $||\mathbf{x}_{t_2} - \mathbf{x}_{t_3}|| \geq d/2$. Additionally, since $\tilde{\beta}^m(\mathbf{x}^{-m}), x^m \in \Delta E^m$ for all m, it follows that $||\dot{x}^m|| = ||\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m|| \leq ||x^m|| + ||\tilde{\beta}^m(\mathbf{x}^{-m})|| \leq 2$ and hence $||\dot{\mathbf{x}}|| \leq \sum_m ||\dot{x}^m|| \leq 2M$. Thus, we obtain $t_3 - t_2 \geq d/4M$, and hence part (i) of Theorem 4.2 implies that

$$\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) \geq (a - M(\delta + \tau))d/4M. \tag{39}$$

Let $\overline{\phi}_k = \max_{\{\mathbf{x} \mid ||\mathbf{x}-\mathbf{x}_k^*|| \leq f(\bar{\epsilon}+M(\delta+\tau))\}} \phi(\mathbf{x})$ and define $\mathbf{y}_k$ as a strategy profile which achieves this maximum. Similarly, let $\underline{\phi}_l = \min_{\{\mathbf{x} \mid ||\mathbf{x}-\mathbf{x}_l^*|| \leq f(\bar{\epsilon}+M(\delta+\tau))\}} \phi(\mathbf{x})$ and define $\mathbf{y}_l$ as a strategy profile which achieves this minimum. Since $||\mathbf{x}_{t_1} - \mathbf{x}_k^*||, ||\mathbf{x}_{t_4} - \mathbf{x}_l^*|| \leq f(\bar{\epsilon}+M(\delta+\tau))$, it follows that $||\mathbf{x}_{t_1} - \mathbf{y}_k||, ||\mathbf{x}_{t_4} - \mathbf{y}_l|| \leq 2f(\bar{\epsilon} + M(\delta + \tau))$. Thus, by Lipschitz continuity of the potential function it follows that $\overline{\phi}_k - \phi(\mathbf{x}_{t_1}) \leq 2f(\bar{\epsilon} + M(\delta + \tau))L$ and $\phi(\mathbf{x}_{t_4}) - \underline{\phi}_l \leq 2f(\bar{\epsilon} + M(\delta + \tau))L$. From these inequalities we conclude that

$$\begin{aligned}
\underline{\phi}_l - \overline{\phi}_k &\geq \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) - 4f(\bar{\epsilon} + M(\delta + \tau))L \\
&\geq \phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) - 4f(\bar{\epsilon} + M(\delta + \tau))L \\
&\geq (a - M(\delta + \tau))d/4M - 4f(\bar{\epsilon} + M(\delta + \tau))L,
\end{aligned} \tag{40}$$

where the last two lines follow from (38), and (39) and $\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) = (\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3})) + (\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2})) + (\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}))$.

Since $\bar{\delta} \geq \delta + \tau$, by step 2 and the fact that f is weakly increasing in its argument we have

$$\begin{aligned}
(a - M(\delta + \tau))d/4M - 4f(\bar{\epsilon} + M(\delta + \tau))L &\geq (a - M\bar{\delta})d/4M - 4f(\bar{\epsilon} + M\bar{\delta})L \\
&= 4L\left((a - M\bar{\delta})d/16ML - f(\bar{\epsilon} + M\bar{\delta})\right) \\
&> 0.
\end{aligned} \tag{41}$$

Thus, (40) and (41) imply that $\underline{\phi}_l - \overline{\phi}_k > 0$. Hence, we conclude that if the trajectory leaves the component of the $\bar{\epsilon} + M(\delta + \tau)$-equilibrium set around $\mathbf{x}_k^*$ and enters to that around equilibrium $\mathbf{x}_l^*$, the maximum potential in the first set is smaller than the minimum potential in the second one. Since this is true for arbitrary equilibria $\mathbf{x}_k^*$ and $\mathbf{x}_l^*$, it follows that once trajectories enter the component of approximate equilibrium set around equilibrium $\mathbf{x}_l^*$, they cannot revisit the component of the approximate equilibrium set around $\mathbf{x}_k^*$. Thus, we conclude that after some time trajectories can visit the $\bar{\epsilon} + M(\delta + \tau)$-equilibrium set around at most one equilibrium.

Step 4: Let ϵ, ϵ_1 be such that $0 < \epsilon_1 < \epsilon \leq \bar{\epsilon}$. By Theorem 4.2 it follows that $\dot{\phi}(\mathbf{x}) \geq \epsilon_1$ for any $\mathbf{x} \notin \mathcal{X}_{\epsilon_1 + M(\delta+\tau)}$. Thus, it follows that after any time instant T, there exists another time instant when the set $\mathcal{X}_{\epsilon_1+M(\delta+\tau)}$ is visited. By Step 3 it follows that after some time the trajectory visits the component of $\epsilon+M(\delta+\tau)$-equilibrium set only around one equilibrium.

Assume that after time instant T, the component of $\epsilon+M(\delta+\tau)$-equilibrium set around only a single equilibrium, say $\mathbf{x}_k^*$, is visited, at $t_1 > t$ the trajectory leaves the $\epsilon_1+M(\delta+\tau)$-equilibrium set, at $t_2 > t_1$ it leaves the $f(M(\delta+\tau)+\epsilon)$ neighborhood of $\mathbf{x}_k^*$, at $t_3 > t_2$, it returns back to this neighborhood and at $t_4 > t_3$ it returns to the $\epsilon_1+M(\delta+\tau)$-equilibrium set around $\mathbf{x}_k^*$. Let d^* be the distance the trajectory gets from the $f(M(\delta+\tau)+\epsilon)$ neighborhood of $\mathbf{x}_k^*$ (note that this set contains the relevant component of the $\epsilon_1+M(\delta+\tau)$-equilibrium set). Since $||\dot{\mathbf{x}}|| \leq 2M$, as shown in the proof of step 3, it follows that $t_3 - t_2 \geq \frac{2d^*}{2M} = \frac{d^*}{M}$. Additionally, since $\dot{\phi}(\mathbf{x}) \geq \epsilon$ outside the $M(\delta+\tau)+\epsilon$ equilibrium set, it follows that potential increases at least by $d^*\epsilon/M$, when trajectory leaves the $f(M(\delta+\tau)+\epsilon)$ neighborhood by d^* and returns back to it, i.e., $\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) \geq d^*\epsilon/M$. Similarly, $\dot{\phi}(\mathbf{x}) \geq \epsilon_1 > 0$ outside the $\epsilon_1+M(\delta+\tau)$-equilibrium set, and hence $\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}), \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3}) \geq 0$. Using these, we obtain

$$\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) \geq d^*\epsilon/M. \tag{42}$$

On the other hand, by definition $\mathbf{x}_{t_4}$ and $\mathbf{x}_{t_1}$ belong to the $\epsilon_1+M(\delta+\tau)$-equilibrium set around $\mathbf{x}_k^*$, and hence $||\mathbf{x}_{t_4} - \mathbf{x}_{t_1}|| \leq 2f(\epsilon_1+M(\delta+\tau))$. Thus, the Lipschitz continuity of the potential function implies that $\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) \leq 2f(\epsilon_1+M(\delta+\tau))L$. Together with (42) this implies that

$$d^*\epsilon/M \leq 2f(\epsilon_1+M(\delta+\tau))L, \tag{43}$$

or equivalently $d^* \leq 2MLf(\epsilon_1+M(\delta+\tau)/\epsilon$. Since the distance between equilibrium $\mathbf{x}_k^*$, and any point in the $f(M(\delta+\tau)+\epsilon)$ neighborhood is at most $f(M(\delta+\tau)+\epsilon)$, it follows that the trajectory can become at most $f(M(\delta+\tau)+\epsilon)+2MLf(\epsilon_1+M(\delta+\tau))/\epsilon$ distant from equilibrium $\mathbf{x}_k^*$. Since this is true for all $\epsilon_1 \in (0,\epsilon)$, taking $\epsilon_1 \to 0$, the result follows from upper semicontinuity of f proved in Step 1.

□