Strategic Form Games and Nash Equilibrium

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Abstract

This article introduces strategic form games, which provide a framework for the analysis of strategic interactions in multi-agent environments. We present the main solution concept in strategic form games, *Nash equilibrium*, and provide tools for its systematic study. We present fundamental results for existence and uniqueness of Nash equilibria and discuss their efficiency properties. We conclude with current research directions in this area.

Keywords: Strategic form games, Nash equilibrium, existence, uniqueness, efficiency.

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1 Introduction

Many problems in communication, decision and technological networks as well as in social and economic situations depend on human choices, which are made in anticipation of the behavior of the others in the system. Examples include how to map your drive over a road network, how to use the communication medium, how to choose strategies for resource use and more conventional economic, financial, and social decisions such as which products to buy, which technologies to invest in or who to trust. The defining feature of all of these interactions is the dependence of an agent's objective (payoff, utility, or survival) on others' actions. Game theory focuses on formal analysis of such strategic interactions. Here we will review strategic form games, which focus on static game-theoretic interactions and present the relevant solution concept.

2 Strategic Form Games

A *strategic form* game is a model for a static game in which all players act simultaneously without knowledge of other players' actions.

Definition 1. *(Strategic Form Game) A strategic form game is a triplet* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *where*

1. $\mathcal{I}$ *is a finite set of players,* $\mathcal{I} = \{1, \ldots, I\}$.

2. S_i *is a non-empty set of available actions for player* i.

3. $u_i : S \to \mathbb{R}$ *is the utility (payoff) function of player* i *where* $S = \prod_{i\in\mathcal{I}} S_i$.

We will use the terms *action* and *(pure) strategy* interchangeably.[1] We denote by $s_i \in S_i$ an action for player i, and by $s_{-i} = [s_j]_{j\neq i}$ a vector of actions for all players *except* i. We refer to the tuple $(s_i, s_{-i}) \in S$ as an *action (strategy) profile* or *outcome.* We also denote by $S_{-i} = \prod_{j\neq i} S_j$ the set of actions (strategies) of all players except i. Our convention throughout will be that each player i is interested in action profiles that "maximize" his utility function u_i.

The next two examples illustrate strategic form games with finite and infinite strategy sets.

Example 1 (Finite Strategy Sets)**.** *We consider a two player game with finite strategy sets. Such a game can be represented in matrix form, where the rows correspond to the actions of player 1 and columns represent the actions of player 2. The cell indexed by row* x *and column* y *contains a pair,* (a, b)*, where* a *is the payoff to player 1 and* b *is the payoff to player 2, i.e.,* $a = u_1(x, y)$ *and* $b = u_2(x, y)$*. This class of games is sometimes referred to as bimatrix games. For example, consider the following game of "Matching Pennies."*

[1] We will later use the term "mixed strategy" to refer to randomizations over actions.

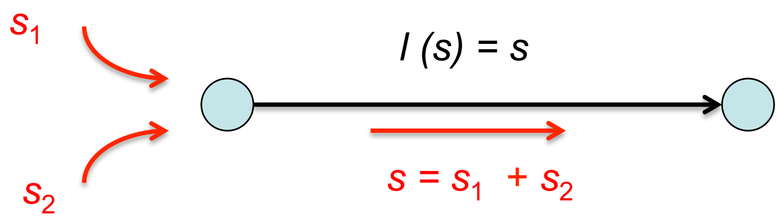

Figure 1: A network game with two players.

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

Matching Pennies.

This game represents "pure conflict" in the sense that one player's utility is the negative of the utility of the other player, i.e., the sum of the utilities for both players at each outcome is "zero." This class of games is referred to as zero-sum games (or constant-sum games) and has been studied extensively in the game theory literature [3].

Example 2 (Infinite Strategy Sets)**.** *We next present a game with infinite strategy sets. We consider a simple network game where two players send data or information flows over a communication network represented by a single link. Each player i derives a value for sending s_i units of flow over the link given by*

$$v_i(s_i) = \begin{cases} a_i s_i - \frac{s_i^2}{2} & \text{if } s_i \leq a_i, \\ \frac{a_i^2}{2} & \text{if } s_i \geq a_i, \end{cases}$$

where $a_i \in [0,1]$ is a player specific scalar. Each player also incurs a per flow delay or latency cost, due to congestion on the link, represented by the function $l(s) = s$, where s is the total flow on the link, i.e., $s = s_1 + s_2$ (see Figure 1). The resulting interactions can be represented by the strategic form game $\langle \mathcal{I}, (S_i), (u_i) \rangle$, which consists of:

1. *A set of two players, $\mathcal{I} = 1, 2$.*
2. *A strategy set $S_i = [0,1]$ for each player i, where $s_i \in S_i$ represents the amount of flow player i sends over the link.*
3. *A utility function u_i for each player i given by value derived from sending s_i units of flow minus the total latency cost, i.e.,*

$$u_i(s_1, s_2) = v_i(s_i) - s_i l(s_1 + s_2).$$

3 Nash Equilibrium

We next introduce the fundamental solution concept for strategic form games, *Nash equilibrium*. A Nash equilibrium captures a steady state of the play in a strategic form game such that each player acts optimally given their "correct" conjectures about the behavior of the other players.

Definition 2 (Nash Equilibrium). *A (pure strategy) Nash equilibrium of a strategic form game* $\langle \mathcal{I}, (S_i), (u_i)_{i\in\mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$*, we have*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

Hence, a Nash equilibrium is a strategy profile s^* such that no player i can profit by unilaterally deviating from his strategy s_i^*, assuming every other player j follows his strategy s_j^*. The definition of a Nash equilibrium can be restated in terms of best-response correspondences.

Definition 3 (Nash Equilibrium - Restated). *Let* $\langle \mathcal{I}, (S_i), (u_i)_{i\in\mathcal{I}} \rangle$ *be a strategic form game. For any* $s_{-i} \in S_{-i}$*, consider the best-response correspondence of player* i*,* $B_i(s_{-i})$*, given by*

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}.$$

We say that an action profile s^* *is a Nash equilibrium if*

$$s_i^* \in B_i(s_{-i}^*) \qquad \text{for all } i \in \mathcal{I}.$$

Thus, if we define the best-response correspondence $B(s) = [B_i(s_{-i})]_{i\in\mathcal{I}}$, the set of Nash equilibria are given by the set of fixed points of $B(s)$. Below we give two examples of games with pure strategy Nash equilibria.

Example 3 (Battle of the Sexes). *Consider a two player game with the following payoff structure:*

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

Battle of the Sexes.

*This game, referred to as the Battle of the Sexes game, represents a scenario in which the two players wish to coordinate their actions, but have different preferences over their actions. This game has two pure strategy Nash equilibria, i.e., the strategy profiles (*Ballet, Ballet*) and (*Soccer, Soccer*).*

Example 4. *Recall the network game given in Example 2. To simplify the computations, let us assume without loss of generality that* $a_1 \geq a_2 \geq \frac{a_1}{3}$*. It can be seen that the best-response functions (single-valued in this case) of the players are given by*

$$B_i(s_{-i}) = \max\left\{0, \frac{a_i - s_{-i}}{3}\right\} \qquad \text{for } i = 1, 2.$$

The unique pure strategy Nash equilibrium of this game is the fixed point of these functions given by

$$(s_1^*, s_2^*) = \left(\frac{3a_1 - a_2}{8}, \frac{3a_2 - a_1}{8}\right).$$

3.1 Mixed Strategy Nash Equilibrium

Consider the two player "Penalty Kick" game between a penalty taker and a goal keeper that has the same payoff structure as the matching pennies:

	LEFT	RIGHT
LEFT	$1, -1$	$-1, 1$
RIGHT	$-1, 1$	$1, -1$

Penalty Kick Game.

This game does not have a pure strategy Nash equilibrium. It can be verified that if the penalty taker (column player) commits to a pure strategy, e.g., chooses LEFT, then the best response of the goal keeper (row player) would be to choose the same side leading to a payoff of -1 for the penalty taker. In fact, the penalty taker would be better off following a strategy which randomizes between LEFT and RIGHT, ensuring that the goal keeper cannot perfectly match his action. This is the idea of "randomized" or mixed strategies which we discuss next.

We first introduce some notation. Let Σ_i denote the set of probability measures over the pure strategy (action) set S_i. We use $\sigma_i \in \Sigma_i$ to denote the *mixed strategy* of player i. When S_i is a finite set, a mixed strategy is a finite dimensional probability vector, i.e., a vector whose elements denote the probability with which a particular action will be played. For example, if S_i has two elements, the set of mixed strategies Σ_i is the one-dimensional probability simplex, i.e., $\Sigma_i = \{(x_1, x_2) \mid x_i \geq 0,\ x_1 + x_2 = 1\}$. We use $\sigma \in \Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$ to denote a *mixed strategy profile*. Note that this implicitly assumes that players randomize independently. We similarly denote $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$.

Following von Neumann-Morgenstern expected utility theory, we extend the payoff functions u_i from S to Σ by

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s),$$

i.e., the payoff of a mixed strategy σ is given by the expected value of pure strategy payoffs under the distribution σ.

We are now ready to define the mixed strategy Nash equilibrium.

Definition 4 (Mixed Strategy Nash Equilibrium)**.** *A mixed strategy profile σ^* is a mixed strategy Nash equilibrium if for each player i,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \qquad \textit{for all } \sigma_i \in \Sigma_i.$$

Note that since $u_i(\sigma_i, \sigma_{-i}^*) = \int_{S_i} u_i(s_i, \sigma_{-i}^*) d\sigma_i(s_i)$, it is sufficient to check only *pure* strategy "deviations" when determining whether a given profile is a Nash equilibrium. This leads to the following characterization of a mixed strategy Nash equilibrium.

Proposition 1. *A mixed strategy profile σ^* is a mixed strategy Nash equilibrium if and only if for each player i,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \qquad \textit{for all } s_i \in S_i.$$

We also have the following useful characterization of a mixed strategy Nash equilibrium in finite strategy set games.

Proposition 2. *Let $G = \langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ be a strategic form game with finite strategy sets. Then, $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$, every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*.*

Proof. Let σ^* be a mixed strategy Nash equilibrium, and let $E_i^* = u_i(\sigma_i^*, \sigma_{-i}^*)$ denote the expected utility for player i. By Proposition 1, we have

$$E_i^* \geq u_i(s_i, \sigma_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

We first show that $E_i^* = u_i(s_i, \sigma_{-i}^*)$ for all s_i in the support of σ_i^* (combined with the preceding relation, this proves one implication). Assume to arrive at a contradiction that this is not the case, i.e., there exists an action s_i' in the support of σ_i^* such that $u_i(s_i', \sigma_{-i}^*) < E_i^*$. Since $u_i(s_i, \sigma_{-i}^*) \leq E_i^*$ for all $s_i \in S_i$, this implies that

$$\sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) < E_i^*,$$

which is a contradiction. The proof of the other implication is similar and is therefore omitted. ■

It follows from this characterization that every action in the support of any player's equilibrium mixed strategy yields the same payoff. This characterization extends to games with infinite strategy sets: $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$, given σ_{-i}^*, no action in S_i yields a payoff that exceeds his equilibrium payoff, and the set of actions that yields a payoff less than his equilibrium payoff has σ_i^*-measure zero.

Example 5. *Let us return to the Battle of the Sexes game.*

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

Battle of the Sexes.

Recall that this game has 2 pure strategy Nash equilibria. Using the characterization result in Proposition 2, we show that it has a unique *mixed strategy Nash equilibrium (which is not a pure strategy Nash equilibrium). First, by using Proposition 2 (and inspecting the payoffs), it can be seen that there are no Nash equilibria where only one of* *the players randomizes over its actions. Now, assume instead that player 1 chooses the action* BALLET *with probability* $p \in (0,1)$ *and* SOCCER *with probability* $1-p$*, and that player 2 chooses* BALLET *with probability* $q \in (0,1)$ *and* SOCCER *with probability* $1-q$*. Using Proposition 2 on player 1's payoffs, we have the following relation*

$$2 \times q + 0 \times (1-q) = 0 \times q + 1 \times (1-q).$$

Similarly, we have

$$1 \times p + 0 \times (1-p) = 0 \times p + 2 \times (1-p).$$

We conclude that the only possible mixed strategy Nash equilibrium is given by $q = \frac{1}{3}$ *and* $p = \frac{2}{3}$.

4 Existence of Nash Equilibrium

The first question that one contemplates in analyzing a strategic form game is whether it has a pure or mixed strategy Nash equilibrium. While it may be possible to explicitly construct a Nash equilibrium (either using computational means or characterization results), this may be a tedious task both in the case of large finite strategy set games or infinite strategy set games with complicated utility functions. One is therefore often interested in establishing existence of an equilibrium, using conditions on the utility functions and constraint sets, before trying to understand its properties. In the sequel, we present results on existence of an equilibrium for games with finite and infinite strategy sets. The proofs of such existence results typically use fixed point arguments on the best response correspondences of the players. They are omitted here and can be found in graduate level game theory text books (see [11], [16]).

4.1 Finite Strategy Set Games

We have seen that while the matching pennies game (and the penalty kick game with the same payoff structure) does not have a pure strategy Nash equilibrium, it has a mixed strategy Nash equilibrium. The next theorem, which is by Nash, states that this existence result extends to all finite strategy set games.

Theorem 1. ***(Nash)*** *Every strategic form game with finite strategy sets has a mixed strategy Nash equilibrium.*

4.2 Infinite Strategy Set Games

A stronger result on existence of a pure strategy Nash equilibrium can be established in infinite strategy set games under some topological conditions on the utility functions and constraint sets (see [9], [10], [12]).

Theorem 2. ***(Debreu, Fan, Glicksberg)*** *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ *with infinite strategy sets such that for each* $i \in \mathcal{I}$*:*

1. S_i is convex and compact.

2. $u_i(s_i, s_{-i})$ is continuous in s_{-i}.

3. $u_i(s_i, s_{-i})$ is continuous and quasiconcave in s_i.[2]

The game has a pure strategy Nash equilibrium.

Note that Theorem 1 is a special case of this result. For games with finite strategy sets, mixed strategy sets are simplices, hence are convex and compact, and utilities are linear in (mixed) strategies, hence they are concave functions of (mixed) strategies (and continuos functions of mixed strategy profiles).

The next example shows that quasiconcavity cannot be dispensed with in the previous existence result.

Example 6. *Consider the game where two players pick a location $s_1, s_2 \in \mathbb{R}^2$ on the circle. The payoffs are*

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2),$$

where $d(s_1, s_2)$ denotes the Euclidean distance between $s_1, s_2 \in \mathbb{R}^2$. It can be verified that this game does not have a pure strategy Nash equilibrium. However, the strategy profile where both players mix uniformly on the circle is a mixed strategy Nash equilibrium.

Without quasiconcavity, one can establish the following existence result (see [12]).

Theorem 3. ***(Glicksberg)*** *Consider a strategic form game $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$, where the S_i are nonempty compact metric spaces, and the $u_i : S \to \mathbb{R}$ are continuous functions. The game has a mixed strategy Nash equilibrium.*

5 Uniqueness of Nash Equilibrium

Another important question that arises in the analysis of strategic form games is whether the Nash equilibrium is unique. This is important for the predictive power of Nash equilibrium since with multiple equilibria, the outcome of the game cannot be uniquely pinned down. The following result by Rosen provides sufficient conditions for uniqueness of an equilibrium in games with infinite strategy sets (see [20]).[3]

We first introduce some notation to state this result. Given a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$, we use the notation $\nabla f(x)$ to denote the gradient vector of f at point x, i.e.,

$$\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}\right]^T.$$

[2]Let X be a convex set. A function $f : X \to \mathbb{R}$ is quasiconcave if every upper level set of the function, i.e., $\{x \in X \mid f(x) \geq \alpha\}$ for every scalar α, is a convex set (see [4]).

[3]Except for games that are strictly dominant solvable, there are no general uniqueness results for finite strategic form games.

Given a scalar-valued function $F : \prod_{i=1}^{I} \mathbb{R}^{m_i} \to \mathbb{R}$, we use the notation $\nabla_i F(x)$ to denote the gradient vector of F with respect to x_i at point x, i.e.,

$$\nabla_i F(x) = \left[\frac{\partial F(x)}{\partial x_i^1}, \ldots, \frac{\partial F(x)}{\partial x_i^{m_i}} \right]^T .$$

We use the notation $\nabla F(x)$ to denote

$$\nabla F(x) = \left[\nabla_1 F_1(x), \ldots, \nabla_I F_I(x) \right]^T . \tag{1}$$

We assume that the strategy set S_i of each player i is given by

$$S_i = \{ x_i \in \mathbb{R}^{m_i} \mid h_i(x_i) \geq 0 \}, \tag{2}$$

where $h_i : \mathbb{R}^{m_i} \mapsto \mathbb{R}$ is a concave function.[4] The next definition introduces the key condition used in establishing the uniqueness of a pure strategy Nash equilibrium.

Definition 5. *We say that the utility functions* $(u_1, \ldots, u_I)$ *are* **diagonally strictly concave** *for* $x \in S$, *if for every* $x^*, \bar{x} \in S$, *we have*

$$(\bar{x} - x^*)^T \nabla u(x^*) + (x^* - \bar{x})^T \nabla u(\bar{x}) > 0.$$

We can now state the result on uniqueness of pure strategy Nash equilibrium in strategic form games.

Theorem 4. ***(Rosen)*** *Consider a strategic form game* $\langle \mathcal{I}, (S_i), (u_i) \rangle$. *For all* $i \in \mathcal{I}$, *assume that the strategy sets* S_i *are given by Eq. (2), where* h_i *is a concave function, and there exists some* $\tilde{x}_i \in \mathbb{R}^{m_i}$ *such that* $h_i(\tilde{x}_i) > 0$. *Assume also that the utility functions* $(u_1, \ldots, u_I)$ *are diagonally strictly concave for* $x \in S$. *Then the game has a unique pure strategy Nash equilibrium.*

We next provide a tractable sufficient condition for the utility functions to be diagonally strictly concave. Let $U(x)$ denote the Jacobian of $\nabla u(x)$ [see Eq. (1)]. Specifically, if the x_i are all 1-dimensional, then $U(x)$ is given by

$$U(x) = \begin{pmatrix} \frac{\partial^2 u_1(x)}{\partial x_1^2} & \frac{\partial^2 u_1(x)}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 u_2(x)}{\partial x_2 \partial x_1} & \ddots & \\ \vdots & & \end{pmatrix} .$$

Proposition 3. ***(Rosen)*** *For all* $i \in \mathcal{I}$, *assume that the strategy sets* S_i *are given by Eq. (2), where* h_i *is a concave function. Assume that the symmetric matrix* $(U(x) + U^T(x))$ *is negative definite for all* $x \in S$, *i.e., for all* $x \in S$, *we have*

$$y^T (U(x) + U^T(x)) y < 0, \qquad \forall\, y \neq 0.$$

Then, the payoff functions $(u_1, \ldots, u_I)$ *are diagonally strictly concave for* $x \in S$.

[4]Since h_i is concave, it follows that the set S_i is a convex set.

Rosen's sufficient conditions for uniqueness are quite strong. Recent work has extended such uniqueness results to hold under weaker conditions using differential topology tools. The main idea is to provide sufficient conditions so that the indices of all stationary points can be shown to be positive, which from a generalization of the Poincare-Hopf theorem ([23], [24]) implies that there exists a unique equilibrium (see [22] for applications of this methodology to several network games).

6 Efficiency of Nash Equilibria

Because the Nash equilibrium corresponds to the fixed point of the best response correspondences of the players, there is no presumption that it is efficient or maximizes any well-defined weighted sum of utility functions of the players. This fact is clearly illustrated by the well-known Prisoner's Dilemma game. For some $a > 0, b > 0, c > 0$ with $a > b$, the payoff matrix is given by:

	Don't Confess	Confess
Don't Confess	a, a	$b - c, a + c$
Confess	$a + c, b - c$	b, b

Prisoner's Dilemma.

This game, generally used for capturing the dilemma of cooperation among selfish agents, has a unique (pure strategy) Nash equilibrium,[5] which is (Confess, Confess). This clearly illustrates two aspects of the inefficiencies that arise in Nash equilibria. First, the unique Nash equilibrium is Pareto inferior meaning that if both players cooperated and chose Don't Confess, they would both obtain the higher payoff of b. Second, the extent of inefficiency can be arbitrarily large based on the values of a and b. We can capture this by the *efficiency loss* (or *Price of Anarchy* as known in the literature) defined as

$$\text{Efficiency Loss} = \inf_{parameters} \frac{\sum_i u_i(\text{equilibrium})}{\sum_i u_i(\text{social optimum})},$$

where the social optimum is the strategy profile that maximizes the sum of utility functions. In the preceding example, this is clearly

$$\inf_{a,b} \frac{b}{a} = 0,$$

showing that efficiency loss can be arbitrarily large. In problems that have more structure, the efficiency loss can be bounded away from zero. A well-known example is by Pigou, which showed that in a network routing game where the congestion penalty can be described by linear latency functions (see Example 2, the efficiency loss is 3/4 [19]. Roughgarden and Tardos in an important contribution [21] showed that this is a lower bound for such routing games over all possible network topologies.

[5] In fact each player has a dominant strategy, see [11].

7 Summary and Future Directions

This article has provided an introduction to the basics of strategic form games. After defining the concept of Nash equilibrium, which is the basis of much of recent game theory, we have presented fundamental results on its existence and uniqueness. We also briefly discussed issues of efficiency of Nash equilibria.

Though game theory is a mature field, there are still several important areas for inquiry. The first is a more systematic analysis and categorization of classes of games by their equilibrium and efficiency properties. Recent work [5], [7], [6] provides tools for systematically analyzing equivalence classes of games that may be useful for such an investigation. The second area that is very much active concerns computational issues, which we have not considered here. Recent literature showed that computation of Nash equilibria in finite strategy set games is potentially hard and focused on developing algorithms for computing approximate Nash equilibria (see [8], [15]). Ongoing research in this area focuses on infinite strategy set games and exploits special structure to develop algorithms for computing (exact and approximate) Nash equilibria ([18], [25]). A third area is to develop a better application of tools of strategic form games and understand the resulting efficiency losses in networks and large scale systems. Work in this area uses game-theoretic models to investigate resource allocation, pricing and investment problems in networks ([14], [2], [1], [17]). A fourth area of research is to develop and apply alternative solution concepts for strategic form games. While some of the research in game theory has focused on subsets of Nash Equilibria (see [11]), from a computational point of view, the set of correlated equilibria, which is a superset of the set of Nash Equilibria, is also attractive since it can be represented as the optimal solution set of a linear program. Correlated equilibrium can be implemented using a correlation scheme (a trusted party) or cryptographic tools as shown in [13]. Recent work investigates alternative solution concepts for symmetric games intermediate between Nash and correlated equilibria [26], which can be implemented using specific correlation schemes.

References

[1] D. Acemoglu, K. Bimpikis, and A. Ozdaglar, *Price and capacity competition*, Games and Economic Behavior **66** (2009), no. 1, 1–26.

[2] D. Acemoglu and A. Ozdaglar, *Competition and efficiency in congested markets*, Mathematics of Operations Research **32** (2007), no. 1, 1–31.

[3] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*, London/New York: Academic Press, 1995.

[4] D. Bertsekas, A. Nedich, and A. Ozdaglar, *Convex analysis and optimization*, Belmont, Massachusetts: Athena Scientific, 2003.

[5] U. O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, *Flows and decompositions of games: harmonic and potential games*, Mathematics of Operations Research **36** (2011), no. 3, 474–503.

[6] U. O. Candogan, A. Ozdaglar, and P. A. Parrilo, *A projection framework for near-potential games*, Proc. of the IEEE Conference on Decision and Control, CDC, 2010.

[7] ______, *Dynamics in near-potential games*, forthcoming, Games and Economic Behavior, 2013.

[8] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou, *The complexity of computing a Nash equilibrium*, Proc. of the 38th ACM Symposium on Theory of Computing, STOC, 2006.

[9] D. Debreu, *A social equilibrium existence theorem*, Proceedings of the National Academy of Sciences **38** (1952), 886–893.

[10] K. Fan, *Fixed point and minimax theorems in locally convex topological linear spaces*, Proceedings of the National Academy of Sciences **38** (1952), 121–126.

[11] D. Fudenberg and J. Tirole, *Game theory*, Cambridge, Massachusetts: MIT Press, 1991.

[12] I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points*, Proceedings of the National Academy of Sciences **38** (1952), 170–174.

[13] S. Izmalkov, M. Lepinski, S. Micali, and A. Shelat, *Transparent computation and correlated equilibrium*, Working paper, 2007.

[14] R. Johari and J.N. Tsitsiklis, *Efficiency loss in a network resource allocation game*, Mathematics of Operations Research **29** (2004), no. 3, 407–435.

[15] R. J. Lipton, E. Markakis, and A. Mehta, *Playing large games using simple strategies*, Proc. of the ACM Conference in Electronic Commerce, EC, 2003.

[16] R. B. Myerson, *Game theory: analysis of conflict*, Cambridge, Massachusetts: Harvard University Press, 1991.

[17] P. Njoroge, A. Ozdaglar, N. Stier-Moses, and G. Weintraub, *Investment in two-sided markets and the net neutrality debate*, Working paper, 2013.

[18] P. A. Parrilo, *Polynomial games and sum of squares optimization*, Proc. of the IEEE Conference on Decision and Control, CDC, 2006.

[19] A. C. Pigou, *The economics of welfare*, London: Macmillan, 1920.

[20] J. B. Rosen, *Existence and uniqueness of equilibrium points for concave N-person games*, Econometrica **33** (1965), no. 3, 520–534.

[21] T. Roughgarden and E. Tardos, *How bad is selfish routing?*, Proc. of the IEEE Symposium on Foundations of Computer Science, FOCS, 2000.

[22] A. Simsek, A. Ozdaglar, and D. Acemoglu, *Uniqueness of generalized equilibrium for box-constrained problems and applications*, Proc. of Allerton Conference on Communication, Control, and Computing, 2005.

[23] ———, *Generalized Poincare-Hopf theorem for compact nonsmooth regions*, Mathematics of Operations Research **32** (2007), no. 1, 193–214.

[24] ———, *Local indices for degenerate variational inequalities*, Mathematics of Operations Research **33** (2008), no. 2, 291–301.

[25] N. Stein, A. Ozdaglar, and P. A. Parrilo, *Separable and low-rank continuous games*, International Journal of Game Theory **37** (2008), no. 4, 475–504.

[26] ———, *Exchangeable equilibria, part I: symmetric bimatrix games*, Working paper, 2013.