

On the $O(1/k)$ Convergence of Asynchronous Distributed Alternating Direction Method of Multipliers*

Ermin Wei[†] and Asuman Ozdaglar[†]

Abstract

We consider a network of agents that are cooperatively solving a global optimization problem, where the objective function is the sum of privately known local objective functions of the agents and the decision variables are coupled via linear constraints. Recent literature focused on special cases of this formulation and studied their distributed solution through either subgradient based methods with $O(1/\sqrt{k})$ rate of convergence (where k is the iteration number) or Alternating Direction Method of Multipliers (ADMM) based methods, which require a synchronous implementation and a globally known order on the agents. In this paper, we present a novel asynchronous ADMM based distributed method for the general formulation and show that it converges at the rate $O(1/k)$.

I. INTRODUCTION

We consider the following optimization problem with a separable objective function and linear constraints:

$$\begin{aligned} \min_{x_i \in X_i, z \in Z} \quad & \sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & Dx + Hz = 0. \end{aligned} \tag{1}$$

Here each $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a (possibly nonsmooth) convex function, X_i and Z are closed convex subsets of \mathbb{R}^n and \mathbb{R}^W , and D and H are matrices of dimensions $W \times nN$ and $W \times W$. The decision variable x is given by the partition $x = [x'_1, \dots, x'_N]' \in \mathbb{R}^{nN}$, where the $x_i \in \mathbb{R}^n$ are components (subvectors)

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[†]Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology

of x . We denote by set X the product of sets X_i , hence the constraint on x can be written compactly as $x \in X$.

Our focus on this formulation is motivated by *distributed multi-agent optimization problems*, which attracted much recent attention in the optimization, control and signal processing communities. Such problems involve resource allocation, information processing, and learning among a set $\{1, \dots, N\}$ of distributed agents connected through a network $G = (V, E)$, where E denotes the set of M undirected edges between the agents. In such applications, each agent has access to a privately known local objective (or cost) function, which represents the negative utility or the loss agent i incurs at the decision variable x . The goal is to collectively solve a global optimization problem¹

$$\begin{aligned} \min \quad & \sum_{i=1}^N f_i(x) \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{2}$$

This problem can be reformulated in the general formulation of (1) by introducing a local copy x_i of the decision variable for each node i and imposing the constraint $x_i = x_j$ for all agents i and j with edge $(i, j) \in E$. Under the assumption that the underlying network is connected, this condition ensures that each of the local copies are equal to each other. Using the edge-node incidence matrix of network G , denoted by $A \in \mathbb{R}^{Mn \times Nn}$, the reformulated problem can be written compactly as²

$$\begin{aligned} \min_{x_i \in X} \quad & \sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & Ax = 0, \end{aligned} \tag{3}$$

where x is the vector $[x_1, x_2, \dots, x_N]'$. We will refer to this formulation as the *edge-based reformulation*

¹The usefulness of formulation (2) can be illustrated by, among other things, machine learning problems described as follows:

$$\min_x \sum_{i=1}^{N-1} l([W_i x - b_i]) + \pi \|x\|_1,$$

where W_i corresponds to the input sample data (and functions thereof), b_i represents the measured outputs, $W_i x - b_i$ indicates the prediction error and l is the loss function on the prediction error. Scalar π is nonnegative and it indicates the penalty parameter on complexity of the model. The widely used Least Absolute Deviation (LAD) formulation, the Least-Absolute Shrinkage and Selection Operator (Lasso) formulation and l_1 regularized formulations can all be represented by the above formulation by varying loss function l and penalty parameter π (see [3] for more details). The above formulation is a special case of the distributed multi-agent optimization problem (2), where $f_i(x) = l(W_i x - b_i)$ for $i = 1, \dots, N - 1$ and $f_N = \pi_2 \|x\|_1$. In applications where the data pairs (W_i, b_i) are collected and maintained by different sensors over a network, the functions f_i are local to each agent and the need for a distributed algorithm arises naturally.

²The edge-node incidence matrix of network G is defined as follows: Each n -row block of matrix A corresponds to an edge in the graph and each n -column block represent a node. The n rows corresponding to the edge $e = (i, j)$ has $I(n \times n)$ in the i^{th} n -column block, $-I(n \times n)$ in the j^{th} n -column block and 0 in the other columns, where $I(n \times n)$ is the identity matrix of dimension n .

of the multi-agent optimization problem. Note that this formulation is a special case of problem (1) with $D = A$, $H = 0$ and $X_i = X$ for all i . Since these problems often lack a centralized processing unit, it is imperative that iterative solutions of problem (2) involve decentralized computations, meaning that each node (processor) performs calculations independently and on the basis of local information available to it and then communicates this information to its neighbors according to the underlying network structure.

Though there have been many important advances in the design of decentralized optimization algorithms for multi-agent optimization problems, several challenges still remain. First, many of these algorithms are based on first-order subgradient methods, which have slow convergence rates (given by $O(1/\sqrt{k})$ where k is the iteration number), making them impractical in many large scale applications. Second, with the exception of a few recent contributions, existing algorithms are synchronous, meaning that computations are simultaneously performed according to some global clock, but this often goes against the highly decentralized nature of the problem, which precludes such global information being available to all nodes.

In this paper, we focus on the more general formulation (1) and propose an asynchronous decentralized algorithm based on the classical Alternating Direction Method of Multipliers (ADMM) (see [6], [12] for comprehensive tutorials). We adopt the following asynchronous implementation for our algorithm: at each iteration k , a random subset Ψ^k of the constraints are selected, which in turn selects the components of x that appear in these constraints. We refer to the selected constraints as *active constraints* and selected components as the *active components (or agents)*. We design an ADMM-type primal-dual algorithm which at each iteration updates the primal variables using partial information about the problem data, in particular using cost functions corresponding to active components and active constraints, and updates the dual variables corresponding to active constraints. In the context of the edge-based reformulated multi-agent optimization problem (3), this corresponds to a fully decentralized and asynchronous implementation in which a subset of the edges are randomly activated (for example according to local clocks associated with those edges) and the agents incident to those edges perform computations on the basis of their local objective functions followed by communication of updated values with neighbors.

Under the assumption that each constraint has a positive probability of being selected and the constraints have a decoupled structure (which is satisfied by reformulations of the distributed multi-agent optimization problem), our first result shows that the (primal) asynchronous iterates generated by this algorithm converge almost surely to an optimal solution. Our proof relies on relating the asynchronous iterates to *full-information* iterates that would be generated by the algorithm that use full information

about the cost functions and constraints at each iteration. In particular, we introduce a weighted norm where the weights are given by the inverse of the probabilities with which the constraints are activated and constructs a Lyapunov function for the asynchronous iterates using this weighted norm. Our second result establishes a *performance guarantee of $O(1/k)$* for this algorithm under a compactness assumption on the constraint sets X and Z , which to our knowledge is faster than the guarantees available in the literature for this problem. More specifically, we show that the expected value of the difference of the objective function value and the optimal value as well as the expected feasibility violation converges to 0 at rate $O(1/k)$.

Our paper is related to a large recent literature on distributed optimization methods for solving the multi-agent optimization problem. Most closely related is a recent stream which presented distributed synchronous ADMM algorithms for solving problem (2) (or specialized versions of it) (see [32], [33], [24], [43], [49]). These papers have demonstrated the excellent computational performance of ADMM algorithms in the context of several signal processing applications. A closely related work in this stream is our recent paper [46], where we considered problem (2) under the general assumption that the f_i are convex. In [46], we presented an ADMM based algorithm which operates by updating the decision variable x in N steps in a synchronous manner using a deterministic cyclic order and showed that it converges at the rate $O(1/k)$. This algorithm however requires a synchronous implementation and a globally known order on the set of agents. The algorithm presented here selects a subset of the components of the decision variable x randomly and updates the variables (x, z) in two steps by first updating the selected components of x and then updating the z variable.

Another strand of this literature uses first-order (sub)gradient methods for solving problem (2). Much of this work builds on the seminal works [2] and [45], which proposed gradient methods that can parallelize computations across multiple processors. The more recent paper [35] introduced a first-order primal subgradient method for solving problem (2) over deterministically varying networks. This method involves each agent maintaining and updating an estimate of the optimal solution by linearly combining a subgradient step along its local cost function with averaging of estimates obtained from his neighbors (also known as a single consensus step).³ Several follow-up papers considered variants of this method for problems with local and global constraints [26], [36] randomly varying networks [29], [30], [31] and random gradient errors [42], [34]. A different distributed algorithm that relies on Nesterov's dual averaging algorithm [37] for static networks has been proposed and analyzed in [11]. Such gradient

³This work is clearly also related to the extensive literature on consensus and cooperative control, where the goal is to design local deterministic or random update rules to achieve global coordination (for deterministic update rules, see [4], [7], [15], [23], [27], [38], [39], [40], [44]; for random update rules, see [1], [5], [9], [14]).

methods typically have a convergence rate of $O(1/\sqrt{k})$. The more recent contribution [25] focuses on a special case of (2) under smoothness assumptions on the cost functions and availability of global information about some problem parameters, and provided gradient algorithms (with multiple consensus steps) which converge at the faster rate of $O(1/k^2)$.

With the exception of [41] and [22], all algorithms provided in the literature are synchronous and assume that computations at all nodes are performed simultaneously according to a global clock. [41] provides an asynchronous subgradient method that uses gossip-type activation and communication between pairs of nodes and shows (under a compactness assumption on the iterates) that the iterates generated by this method converge almost surely to an optimal solution. The recent independent paper [22] provides an asynchronous randomized ADMM algorithm for solving problem (2) and establishes convergence of the iterates to an optimal solution by studying the convergence of randomized Gauss-Seidel iterations on non-expansive operators. Our paper instead proposes an asynchronous ADMM algorithm for the more general problem (1) and uses a Lyapunov function argument for establishing $O(1/k)$ rate of convergence.

Our algorithm and analysis also build on and combines ideas from several important contributions in the study of ADMM algorithms. Earlier work in this area focuses on the case $C = 2$, where C refers to the number of sequential primal updates at each iteration, and studies convergence in the context of finding zeros of the sum of two maximal monotone operators (more specifically, the Douglas-Rachford operator), see [10], [13], [28]. The recent contribution [20] considered solving problem (1) (with $C = 2$) with ADMM and showed that the objective function values of the iterates converge at the rate $O(1/k)$. Other recent works analyzed the rate of convergence of ADMM and other related algorithms under smoothness conditions on the objective function (see [8], [16], [17]). Another paper [19] considered the case $C \geq 2$ and showed that the resulting ADMM algorithm converges under the more restrictive assumption that each f_i is strongly convex. The recent paper [21] focused on the general case $C \geq 2$ and established a global linear convergence rate using an error bound condition that estimates the distance from the dual optimal solution set in terms of norm of a proximal residual.

The paper is organized as follows: we start in Section II by highlighting the main ideas of the standard ADMM algorithm. In Section III, we focus on the more general formulation (1), present the asynchronous ADMM algorithm and apply this algorithm to solve problem (2) in a distributed way. Section IV contains our convergence and rate of convergence analysis. Section V concludes with closing remarks.

Basic Notation and Notions:

A vector is viewed as a column vector. For a matrix A , we write $[A]_i$ to denote the i^{th} column of

matrix A , and $[A]^j$ to denote the j^{th} row of matrix A . For a vector x , x_i denotes the i^{th} component of the vector. For a vector x in \mathbb{R}^n and set S a subset of $\{1, \dots, n\}$, we denote by $[x]_S$ a vector in \mathbb{R}^n , which places zeros for all components of x outside set S , i.e.,

$$[[x]_S]_i = \begin{cases} x_i & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We use x' and A' to denote the transpose of a vector x and a matrix A respectively. We use standard Euclidean norm (i.e., 2-norm) unless otherwise noted, i.e., for a vector x in \mathbb{R}^n , $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

II. PRELIMINARIES: STANDARD ADMM ALGORITHM

The standard ADMM algorithm solves a separable convex optimization problem where the decision vector decomposes into two variables and the objective function is the sum of convex functions over these variables that are coupled through a linear constraint:⁴

$$\begin{aligned} \min_{x \in X, z \in Z} \quad & F_s(x) + G_s(z) \\ \text{s.t.} \quad & D_s x + H_s z = c, \end{aligned} \tag{4}$$

where $F_s : \mathbb{R}^n \rightarrow \mathbb{R}$ and $G_s : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex functions, X and Z are nonempty closed convex subsets of \mathbb{R}^n and \mathbb{R}^m , and D_s and H_s are matrices of dimension $w \times n$ and $w \times m$.

We consider the augmented Lagrangian function of problem (4) obtained by adding a quadratic penalty for feasibility violation to the Lagrangian function:

$$L_\beta(x, z, p) = F_s(x) + G_s(z) - p'(D_s x + H_s z - c) + \frac{\beta}{2} \|D_s x + H_s z - c\|^2, \tag{5}$$

where p in \mathbb{R}^w is the Lagrange multiplier corresponding to the constraint $D_s x + H_s z = c$ and β is a positive penalty parameter.

The standard ADMM algorithm is an iterative primal-dual algorithm, which can be viewed as an approximate version of the classical augmented Lagrangian method for solving problem (4). It proceeds by approximately minimizing the augmented Lagrangian function through updating the primal variables x and z sequentially within a single pass block coordinate descent (in a Gauss-Seidel manner) at the current Lagrange multiplier (or the dual variable) followed by updating the dual variable through a gradient ascent method (see [21] and [12]). More specifically, starting from some initial vector

⁴Interested readers can find more details in [6] and [12].

(x^0, z^0, p^0) ,⁵ at iteration $k \geq 0$, the variables are updated as

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} L_\beta(x, z^k, p^k), \quad (6)$$

$$z^{k+1} \in \underset{z \in Z}{\operatorname{argmin}} L_\beta(x^{k+1}, z, p^k), \quad (7)$$

$$p^{k+1} = p^k - \beta(D_s x^{k+1} + H_s z^{k+1} - c). \quad (8)$$

We assume that the minimizers in steps (6) and (7) exist, however they need not be unique. Note that the stepsize used in updating the dual variable is the same as the penalty parameter β .

The ADMM algorithm takes advantage of the separable structure of problem (4) and decouples the minimization of functions F_s and G_s since the sequential minimization over x and z involves (quadratic perturbations) these functions separately. This is particularly useful in applications where the minimization over these component functions admit simple solutions and can be implemented in a parallel or decentralized manner.

The analysis of the ADMM algorithm adopts the following standard assumption on problem (4).

Assumption 1: (Existence of a Saddle Point) The Lagrangian function of problem (4) given by

$$L(x, z, p) = F_s(x) + G_s(z) - p'(D_s x + H_s z),$$

has a saddle point, i.e., there exists a solution-multiplier pair (x^*, z^*, p^*) with

$$L(x^*, z^*, p) \leq L(x^*, z^*, p^*) \leq L(x, z, p^*),$$

for all x in \mathbb{R}^n , z in \mathbb{R}^m and p in \mathbb{R}^w .

Note that the existence of a saddle point is equivalent to the existence of a primal dual optimal solution pair. It is well-known that under the given assumptions, the objective function value of the primal sequence $\{x^k, z^k\}$ generated by (6)-(7) converges to the optimal value of problem (4) and the dual sequence $\{p^k\}$ generated by (8) converges to a dual optimal solution (see Section 3.2 of [6]).

III. ASYNCHRONOUS ADMM ALGORITHM

Extending the standard ADMM, we present in this section an asynchronous distributed ADMM algorithm. We present the problem formulation and assumptions in Section III-A. In Section III-B, we discuss the asynchronous implementation considered in the rest of this paper that involves updating a subset of components of the decision vector at each time using partial information about problem data

⁵We use superscripts to denote the iteration number.

and without need for a global coordinator. Section III-C contains the details of the asynchronous ADMM algorithm. In Section III-D, we apply the asynchronous ADMM algorithm to solve the distributed multi-agent optimization problem (2).

A. Problem Formulation and Assumptions

We consider the optimization problem given in (1), which is restated here for convenience:

$$\begin{aligned} \min_{x_i \in X_i, z \in Z} \quad & \sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & Dx + Hz = 0. \end{aligned}$$

This problem formulation arises in large-scale multi-agent (or processor) environments where problem data is distributed across N agents, i.e., each agent has access only to the component function f_i and maintains the decision variable component x_i . The constraints usually represent the coupling across components of the decision variable imposed by the underlying connectivity among the agents. Motivated by such applications, we will refer to each component function f_i as the *local objective function* and use the notation $F : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ to denote the *global objective function* given by their sum:

$$F(x) = \sum_{i=1}^N f_i(x_i). \quad (9)$$

Similar to the standard ADMM formulation, we adopt the following assumption.

Assumption 2: (Existence of a Saddle Point) The Lagrangian function of problem (1),

$$L(x, z, p) = F(x) - p'(Dx + Hz), \quad (10)$$

has a saddle point, i.e., there exists a solution-multiplier pair (x^*, z^*, p^*) with

$$L(x^*, z^*, p) \leq L(x^*, z^*, p^*) \leq L(x, z, p^*) \quad (11)$$

for all x in X , z in Z and p in \mathbb{R}^W .

Moreover, we assume that the matrices have special structure that enables solving problem (1) in an asynchronous manner:

Assumption 3: (Decoupled Constraints) Matrix H is diagonal and invertible. Each row of matrix D has exactly one nonzero element and matrix D has no columns of all zeros.⁶

⁶We assume without loss of generality that each x_i is involved at least in one of the constraints, otherwise, we could remove it from the problem and optimize it separately. Similarly, the diagonal elements of matrix H are assumed to be non-zero, otherwise, that component of variable z can be dropped from the optimization problem.

The diagonal structure of matrix H implies that each component of vector z appears in exactly one linear constraint. The conditions that each row of matrix D has only one nonzero element and matrix D has no column of zeros guarantee the columns of matrix D are linearly independent and hence matrix $D'D$ is positive definite. The condition on matrix D implies that each row of the constraint $Dx + Hz = 0$ involves exactly one x_i . We will see in Section III-D that this assumption is satisfied by the distributed multi-agent optimization problem that motivates this work.

B. Asynchronous Algorithm Implementation

In the large scale multi-agent applications described above, it is essential that the iterative solution of the problem involves computations performed by agents in a decentralized manner (with access to local information) with as little coordination as possible. This necessitates an asynchronous implementation in which some of the agents become active (randomly) in time and update the relevant components of the decision variable using partial and local information about problem data while keeping the rest of the components of the decision variable unchanged. This removes the need for a centralized coordinator or global clock, which is an unrealistic requirement in such decentralized environments.

To describe the asynchronous algorithm implementation we consider in this paper more formally, we first introduce some notation. We call a partition of the set $\{1, \dots, W\}$ a *proper partition* if it has the property that if z_i and z_j are coupled in the constraint set Z , i.e., value of z_i affects the constraint on z_j for any z in set Z , then i and j belong to the same partition, i.e., $\{i, j\} \subset \psi$ for some ψ in the partition. We let Π be a proper partition of the set $\{1, \dots, W\}$, which forms a partition of the set of W rows of the linear constraint $Dx + Hz = 0$. For each ψ in Π , we define $\Phi(\psi)$ to be the set of indices i , where x_i appears in the linear constraints in set ψ . Note that $\Phi(\psi)$ is an element of the power set $2^{\{1, \dots, N\}}$.

At each iteration of the asynchronous algorithm, two random variables Φ^k and Ψ^k are realized. While the pair (Φ^k, Ψ^k) is correlated for each iteration k , these variables are assumed to be independent and identically distributed across iterations. At each iteration k , first the random variable Ψ^k is realized. The realized value, denoted by ψ^k , is an element of the proper partition Π and selects a subset of the linear constraints $Dx + Hz = 0$. The random variable Φ^k then takes the realized value $\phi^k = \Phi(\psi^k)$. We can view this process as activating a subset of the coupling constraints and the components that are involved in these constraints. If $l \in \psi^k$, we say constraint l as well as its associated dual variable p_l is *active* at iteration k . Moreover, if $i \in \Phi(\psi^k)$, we say that component i or agent i is *active* at iteration k . We use the notation $\bar{\phi}^k$ to denote the complement of set ϕ^k in set $\{1, \dots, N\}$ and similarly $\bar{\psi}^k$ to denote the complement of set ψ^k in set $\{1, \dots, W\}$.

Our goal is to design an algorithm in which at each iteration k , only active components of the

decision variable and active dual variables are updated using local cost functions of active agents and active constraints. To that end, we define $f^k : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ as the sum of the local objective functions whose indices are in the subset ϕ^k :

$$f^k(x) = \sum_{i \in \phi^k} f_i(x_i),$$

We denote by D_i the matrix in $\mathbb{R}^{W \times nN}$ that picks up the columns corresponding to x_i from matrix D and has zeros elsewhere. Similarly, we denote by H_l the diagonal matrix in $\mathbb{R}^{W \times W}$ which picks up the element in the l^{th} diagonal position from matrix H and has zeros elsewhere. Using this notation, we define the matrices

$$D_{\phi^k} = \sum_{i \in \phi^k} D_i, \quad \text{and} \quad H_{\psi^k} = \sum_{l \in \psi^k} H_l.$$

We impose the following condition on the asynchronous algorithm.

Assumption 4: (Infinitely Often Update) For all k and all ψ in the proper partition Π ,

$$\mathbb{P}(\Psi^k = \psi) > 0.$$

This assumption ensures that each element of the partition Π is active infinitely often with probability 1. Since matrix D has no columns of all zeros, each of the x_i is involved in some constraints, and hence $\cup_{\psi \in \Pi} \Phi(\psi) = \{1, \dots, N\}$. The preceding assumption therefore implies that each agent i belongs to at least one set $\Phi(\psi)$ and therefore is active infinitely often with probability 1. From definition of the partition Π , we have $\cup_{\psi \in \Pi} \psi = \{1, \dots, W\}$. Thus, each constraint l is active infinitely often with probability 1.

C. Asynchronous ADMM Algorithm

We next describe the asynchronous ADMM algorithm for solving problem (1).

I. Asynchronous ADMM algorithm:

A Initialization: choose some arbitrary x^0 in X , z^0 in Z and $p^0 = 0$.

B At iteration k , random variables Φ^k and Ψ^k takes realizations ϕ^k and ψ^k . Function f^k and matrices D_{ϕ^k} , H_{ψ^k} are generated accordingly.

a The primal variable x is updated as

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} f^k(x) - (p^k)' D_{\phi^k} x + \frac{\beta}{2} \|D_{\phi^k} x + H_{\psi^k} z^k\|^2. \quad (12)$$

with $x_i^{k+1} = x_i^k$, for i in $\bar{\phi}^k$.

b The primal variable z is updated as

$$z^{k+1} \in \underset{z \in Z}{\operatorname{argmin}} -(p^k)' H_{\psi^k} z + \frac{\beta}{2} \|H_{\psi^k} z + D_{\phi^k} x^{k+1}\|^2. \quad (13)$$

with $z_i^{k+1} = z_i^k$, for i in $\bar{\psi}^k$.

c The dual variable p is updated as

$$p^{k+1} = p^k - \beta [D_{\phi^k} x^{k+1} + H_{\psi^k} z^{k+1}]_{\psi^k}. \quad (14)$$

We assume that the minimizers in updates (12) and (13) exist, but need not be unique.⁷ The term $\frac{\beta}{2} \|D_{\phi^k} x + H z^k\|^2$ in the objective function of the minimization problem in update (12) can be written as

$$\frac{\beta}{2} \|D_{\phi^k} x + H z^k\|^2 = \frac{\beta}{2} \|D_{\phi^k} x\|^2 + \beta (H z^k)' D_{\phi^k} x + \frac{\beta}{2} \|H z^k\|^2,$$

where the last term is independent of the decision variable x and thus can be dropped from the objective function. Therefore, the primal x update can be written as

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} f^k(x) - (p^k - \beta H z^k)' D_{\phi^k} x + \frac{\beta}{2} \|D_{\phi^k} x\|^2. \quad (15)$$

Similarly, the term $\frac{\beta}{2} \|H_{\psi^k} z + D_{\phi^k} x^{k+1}\|^2$ in update (13) can be expressed equivalently as

$$\frac{\beta}{2} \|H_{\psi^k} z + D_{\phi^k} x^{k+1}\|^2 = \frac{\beta}{2} \|H_{\psi^k} z\|^2 + \beta (D_{\phi^k} x^{k+1})' H_{\psi^k} z + \frac{\beta}{2} \|D_{\phi^k} x^{k+1}\|^2.$$

We can drop the term $\frac{\beta}{2} \|D_{\phi^k} x^{k+1}\|^2$, which is constant in z , and write update (13) as

$$z^{k+1} \in \underset{z \in Z}{\operatorname{argmin}} -(p^k - \beta D_{\phi^k} x^{k+1})' H_{\psi^k} z + \frac{\beta}{2} \|H_{\psi^k} z\|^2, \quad (16)$$

The updates (15) and (16) make the dependence on the decision variables x and z more explicit and therefore will be used in the convergence analysis. We refer to (15) and (16) as the *primal x and z update* respectively, and (14) as the *dual update*.

⁷Note that the optimization in (12) and (13) are independent of components of x not in ϕ^k and components of z not in ψ^k and thus the restriction of $x_i^{k+1} = x_i^k$, for i not in ϕ^k and $z_i^{k+1} = z_i^k$, for i not in ψ^k still preserves optimality of x^{k+1} and z^{k+1} with respect to the optimization problems in update (12) and (13).

D. Special Case: Distributed Multi-agent Optimization

We apply the asynchronous ADMM algorithm to the edge-based reformulation of the multi-agent optimization problem (3).⁸ Note that each constraint of this problem takes the form $x_i = x_j$ for agents i and j with $(i, j) \in E$. Therefore, this formulation does not satisfy Assumption 3.

We next introduce another reformulation of this problem, used also in Example 4.4 of Section 3.4 in [2], so that each each constraint only involves one component of the decision variable.⁹ More specifically, we let $N(e)$ denote the agents which are the endpoints of edge e and introduce a variable $z = [z_{eq}]_{\substack{e=1, \dots, M \\ q \in N(e)}}$ of dimension $2M$, one for each endpoint of each edge. Using this variable, we can write the constraint $x_i = x_j$ for each edge $e = (i, j)$ as

$$x_i = z_{ei}, \quad -x_j = z_{ej}, \quad z_{ei} + z_{ej} = 0.$$

The variables z_{ei} can be viewed as an estimate of the component x_j which is known by node i . The transformed problem can be written compactly as

$$\begin{aligned} \min_{x_i \in X, z \in Z} \quad & \sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & A_{ei}x_i = z_{ei}, \quad e = 1, \dots, M, i \in \mathcal{N}(e), \end{aligned} \tag{17}$$

where Z is the set $\{z \in \mathbb{R}^{2M} \mid \sum_{q \in N(e)} z_{eq} = 0, e = 1, \dots, M\}$ and A_{ei} denotes the entry in the e^{th} row and i^{th} column of matrix A , which is either 1 or -1 . This formulation is in the form of problem (1) with matrix $H = -I$, where I is the identity matrix of dimension $2M \times 2M$. Matrix D is of dimension $2M \times N$, where each row contains exactly one entry of 1 or -1 . In view of the fact that each node is incident to at least one edge, matrix D has no column of all zeros. Hence Assumption 3 is satisfied.

One natural implementation of the asynchronous algorithm is to associate with each edge an independent Poisson clock with identical rates across the edges. At iteration k , if the clock corresponding to edge (i, j) ticks, then $\phi^k = \{i, j\}$ and ψ^k picks the rows in the constraint associated with edge (i, j) , i.e., the constraints $x_i = z_{ei}$ and $-x_j = z_{ej}$.¹⁰

We associate a dual variable p_{ei} in \mathbb{R} to each of the constraint $A_{ei}x_i = z_{ei}$, and denote the vector of dual variables by p . The primal z update and the dual update [Eqs. (13) and (14)] for this problem are

⁸For simplifying the exposition, we assume $n = 1$ and note that the results extend to $n > 1$.

⁹Note that this reformulation can be applied to any problem with a separable objective function and linear constraints to turn it into a problem of form (1) that satisfies Assumption 3.

¹⁰Note that this selection is a proper partition of the constraints since the set Z couples only the variables z_{ek} for the endpoints of an edge e .

given by

$$\begin{aligned}
z_{ei}^{k+1}, z_{ej}^{k+1} = & \underset{z_{ei}, z_{ej}, z_{ei}+z_{ej}=0}{\operatorname{argmin}} - (p_{ei}^k)'(A_{ei}x_i^{k+1} - z_{ei}) - (p_{ej}^k)'(A_{ej}x_j^{k+1} - z_{ej}) \\
& + \frac{\beta}{2} \left(\|A_{ei}x_i^{k+1} - z_{ei}\|^2 + \|A_{ej}x_j^{k+1} - z_{ej}\|^2 \right), \\
p_{eq}^{k+1} = & p_{eq}^k - \beta(A_{eq}x_q^{k+1} - z_{eq}^{k+1}) \quad \text{for } q = i, j.
\end{aligned} \tag{18}$$

The primal z update involves a quadratic optimization problem with linear constraints which can be solved in closed form. In particular, using first order optimality conditions, we conclude

$$z_{ei}^{k+1} = \frac{1}{\beta}(-p_{ei}^k - v^{k+1}) + A_{ei}x_i^{k+1}, \quad z_{ej}^{k+1} = \frac{1}{\beta}(-p_{ej}^k - v^{k+1}) + A_{ej}x_j^{k+1}, \tag{19}$$

where v^{k+1} is the Lagrange multiplier associated with the constraint $z_{ei} + z_{ej} = 0$ and is given by

$$v^{k+1} = \frac{1}{2}(-p_{ei}^k - p_{ej}^k) + \frac{\beta}{2}(A_{ei}x_i^{k+1} + A_{ej}x_j^{k+1}). \tag{20}$$

Combining these steps yields the following asynchronous algorithm for problem (3) which can be implemented in a decentralized manner by each node i at each iteration k having access to only his local objective function f_i , adjacency matrix entries A_{ei} , and his local variables x_i^k , z_{ei}^k , and p_{ei}^k while exchanging information with one of his neighbors.¹¹

II. Asynchronous Edge Based ADMM algorithm:

A Initialization: choose some arbitrary x_i^0 in X and z^0 in Z , which are not necessarily all equal.

Initialize $p_{ei}^0 = 0$ for all edges e and end points i .

B At time step k , the local clock associated with edge $e = (i, j)$ ticks,

a Agents i and j update their estimates x_i^k and x_j^k simultaneously as

$$x_q^{k+1} = \underset{x_q \in X}{\operatorname{argmin}} f_q(x_q) - (p_{eq}^k)'A_{eq}x_q + \frac{\beta}{2} \|A_{eq}x_q - z_{eq}^k\|^2$$

for $q = i, j$. The updated value of x_i^{k+1} and x_j^{k+1} are exchanged over the edge e .

b Agents i and j exchange their current dual variables p_{ei}^k and p_{ej}^k over the edge e . For $q = i, j$,

¹¹The asynchronous ADMM algorithm can also be applied to a node-based reformulation of problem (2), where we impose the local copy of each node to be equal to the average of that of its neighbors. This leads to another asynchronous distributed algorithm with a different communication structure in which each node at each iteration broadcasts its local variables to all his neighbors, see [47] for more details.

agents i and j use the obtained values to compute the variable v^{k+1} as Eq. (20), i.e.,

$$v^{k+1} = \frac{1}{2}(-p_{ei}^k - p_{ej}^k) + \frac{\beta}{2}(A_{ei}x_i^{k+1} + A_{ej}x_j^{k+1}).$$

and update their estimates z_{ei}^k and z_{ej}^k according to Eq. (19), i.e.,

$$z_{eq}^{k+1} = \frac{1}{\beta}(-p_{eq}^k - v^{k+1}) + A_{eq}x_q^{k+1}.$$

c Agents i and j update the dual variables p_{ei}^{k+1} and p_{ej}^{k+1} as

$$p_{eq}^{k+1} = -v^{k+1} \quad \text{for } q = i, j.$$

d All other agents keep the same variables as the previous time.

IV. CONVERGENCE ANALYSIS FOR ASYNCHRONOUS ADMM ALGORITHM

In this section, we study the convergence behavior of the asynchronous ADMM algorithm under Assumptions 2-4. We show that the primal iterates $\{x^k, z^k\}$ generated by (15) and (16) converge almost surely to an optimal solution of problem (1). Under the additional assumption that the constraint sets X and Z are compact, we further show that the corresponding objective function values converge to the optimal value in expectation at rate $O(1/k)$.

We first recall the relationship between the sets ϕ^k and ψ^k for a particular iteration k , which plays an important role in the analysis. Since the set of active components at time k , ϕ^k , represents all components of the decision variable that appear in the active constraints defined by the set ψ^k , we can write

$$[Dx]_{\psi^k} = [D_{\phi^k}x]_{\psi^k}. \quad (21)$$

We next consider a sequence $\{y^k, v^k, \mu^k\}$, which is formed of iterates defined by a “full information” version of the ADMM algorithm in which all constraints (and therefore all components) are active at each iteration. We will show that under the Decoupled Constraints Assumption (cf. Assumption 3), the iterates generated by the asynchronous algorithm (x^k, z^k, p^k) take the values of (y^k, v^k, μ^k) over the sets of active components and constraints and remain at their previous values otherwise. This association enables us to perform the convergence analysis using the sequence $\{y^k, v^k, \mu^k\}$ and then translate the results into bounds on the objective function value improvement along the sequence $\{x^k, z^k, p^k\}$.

More specifically, at iteration k , we define y^{k+1} by

$$y^{k+1} \in \underset{y \in X}{\operatorname{argmin}} F(y) - (p^k - \beta H z^k)' D y + \frac{\beta}{2} \|D y\|^2. \quad (22)$$

Due to the fact that each row of matrix D has only one nonzero element [cf. Assumption 3], the norm $\|D y\|^2$ can be decomposed as $\sum_{i=1}^N \|D_i y_i\|^2$, where recall that D_i is the matrix that picks up the columns corresponding to component x_i and is equal to zero otherwise. Thus, the preceding optimization problem can be written as a separable optimization problem over the variables y_i :

$$y^{k+1} \in \sum_{i=1}^N \underset{y_i \in X_i}{\operatorname{argmin}} f_i(y_i) - (p^k - \beta H z^k)' D_i y_i + \frac{\beta}{2} \|D_i y_i\|^2.$$

Since $f^k(x) = \sum_{i \in \phi^k} f_i(x_i)$, and $D_{\phi^k} = \sum_{i \in \phi^k} D_i$, the minimization problem that defines the iterate x^{k+1} [cf. Eq. (15)] similarly decomposes over the variables x_i for $i \in \Phi^k$. Hence, the iterates x^{k+1} and y^{k+1} are identical over the components in set ϕ^k , i.e., $[x^{k+1}]_{\phi^k} = [y^{k+1}]_{\phi^k}$. Using the definition of matrix D_{ϕ^k} , i.e., $D_{\phi^k} = \sum_{i \in \phi^k} D_i$, this implies the following relation:

$$D_{\phi^k} x^{k+1} = D_{\phi^k} y^{k+1}. \quad (23)$$

The rest of the components of the iterate x^{k+1} by definition remain at their previous value, i.e., $[x^{k+1}]_{\bar{\phi}^k} = [x^k]_{\bar{\phi}^k}$.

Similarly, we define vector v^{k+1} in Z by

$$v^{k+1} \in \underset{v \in Z}{\operatorname{argmin}} -(p^k - \beta D y^{k+1})' H v + \frac{\beta}{2} \|H v\|^2. \quad (24)$$

Using the diagonal structure of matrix H [cf. Assumption 3] and the fact that Π is a proper partition of the constraint set [cf. Section III-B], this problem can also be decomposed in the following way:

$$v^{k+1} \in \underset{v, [v]_{\psi} \in Z_{\psi}}{\operatorname{argmin}} \sum_{\psi \in \Pi} -(p^k - \beta D y^{k+1})' H_{\psi} [v]_{\psi} + \frac{\beta}{2} \|H_{\psi} [v]_{\psi}\|^2,$$

where H_{ψ} is a diagonal matrix that contains the l^{th} diagonal element of the diagonal matrix H for l in set ψ (and has zeros elsewhere) and set Z_{ψ} is the projection of set Z on component $[v]_{\psi}$. Since the diagonal matrix H_{ψ^k} has nonzero elements only on the l^{th} element of the diagonal with $l \in \psi^k$, the update of $[v]_{\psi}$ is independent of the other components, hence we can express the update on the components of v^{k+1} in set ψ^k as

$$[v^{k+1}]_{\psi^k} \in \underset{v \in Z}{\operatorname{argmin}} -(p^k - \beta D_{\psi^k} x^{k+1})' H_{\psi^k} z + \frac{\beta}{2} \|H_{\psi^k} z\|^2.$$

By the primal z update [cf. Eq. (16)], this shows that $[z^{k+1}]_{\psi^k} = [v^{k+1}]_{\psi^k}$. By definition, the rest of the components of z^{k+1} remain at their previous values, i.e., $[z^{k+1}]_{\bar{\psi}^k} = [z^k]_{\bar{\psi}^k}$.

Finally, we define vector μ^{k+1} in \mathbb{R}^W by

$$\mu^{k+1} = p^k - \beta(Dy^{k+1} + Hv^{k+1}). \quad (25)$$

We relate this vector to the dual variable p^{k+1} using the dual update [cf. Eq. (14)]. We also have

$$[D_{\phi^k} x^{k+1}]_{\psi^k} = [D_{\phi^k} y^{k+1}]_{\psi^k} = [Dy^{k+1}]_{\psi^k},$$

where the first equality follows from Eq. (23) and second is derived from Eq. (21). Moreover, since H is diagonal, we have $[H_{\psi^k} z^{k+1}]_{\psi^k} = [Hv^{k+1}]_{\psi^k}$. Thus, we obtain $[p^{k+1}]_{\psi^k} = [\mu^{k+1}]_{\psi^k}$ and $[p^{k+1}]_{\bar{\psi}^k} = [p^k]_{\bar{\psi}^k}$.

A key term in our analysis will be the *residual* defined at a given primal vector (y, v) by

$$r = Dy + Hv. \quad (26)$$

The residual term is important since its value at the primal vector (y^{k+1}, v^{k+1}) specifies the update direction for the dual vector μ^{k+1} [cf. Eq. (25)]. We will denote the residual at the primal vector (y^{k+1}, v^{k+1}) by

$$r^{k+1} = Dy^{k+1} + Hv^{k+1}. \quad (27)$$

A. Preliminaries

We proceed to the convergence analysis of the asynchronous algorithm. We first present some preliminary results which will be used later to establish convergence properties of asynchronous algorithm. In particular, we provide bounds on the difference of the objective function value of the vector y^k from the optimal value, the distance between μ^k and an optimal dual solution and distance between v^k and an optimal solution z^* . We also provide a set of sufficient conditions for a limit point of the sequence $\{x^k, z^k, p^k\}$ to be a saddle point of the Lagrangian function. The results of this section are independent of the probability distributions of the random variables Φ^k and Ψ^k . Due to space constraints, the proofs of the results of in this section are omitted. We refer the reader to [47] for the missing details.

The next lemma establishes primal feasibility (or zero residual property) of a saddle point of the Lagrangian function of problem (1).

Lemma 4.1: Let (x^*, z^*, p^*) be a saddle point of the Lagrangian function defined as in Eq. (10) of problem (1). Then

$$Dx^* + Hz^* = 0. \quad (28)$$

The next theorem provides bounds on two key quantities, $F(y^{k+1}) - \mu' r^{k+1}$ and $\frac{1}{2\beta} \|\mu^{k+1} - p^*\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^*)\|^2$. These quantities will be related to the iterates generated by the asynchronous ADMM algorithm via a weighted norm and a weighted Lagrangian function in Section IV-B. The weighted version of the quantity $\frac{1}{2\beta} \|\mu^{k+1} - p^*\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^*)\|^2$ is used to show almost sure convergence of the algorithm and the quantity $F(y^{k+1}) - \mu' r^{k+1}$ is used in the convergence rate analysis.

Theorem 4.2: Let $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm (12)-(14). Let $\{y^k, v^k, \mu^k\}$ be the sequence defined in Eqs. (22)-(25) and (x^*, z^*, p^*) be a saddle point of the Lagrangian function of problem (1). The following hold at each iteration k :

$$F(x^*) - F(y^{k+1}) + \mu' r^{k+1} \geq \frac{1}{2\beta} \left(\|\mu^{k+1} - \mu\|^2 - \|p^k - \mu\|^2 \right) \quad (29)$$

$$+ \frac{\beta}{2} \left(\|H(v^{k+1} - z^*)\|^2 - \|H(z^k - z^*)\|^2 \right) + \frac{\beta}{2} \|r^{k+1}\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2,$$

for all μ in \mathbb{R}^W , and

$$0 \geq \frac{1}{2\beta} \left(\|\mu^{k+1} - p^*\|^2 - \|p^k - p^*\|^2 \right) + \frac{\beta}{2} \left(\|H(v^{k+1} - z^*)\|^2 - \|H(z^k - z^*)\|^2 \right) \quad (30)$$

$$+ \frac{\beta}{2} \|r^{k+1}\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2.$$

The following lemma analyzes the limiting properties of the sequence $\{x^k, z^k, p^k\}$. The results will later be used in Lemma 4.5, which provides a set of sufficient conditions for a limit point of the sequence $\{x^k, z^k, p^k\}$ to be a saddle point.

Lemma 4.3: Let $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm (12)-(14). Let $\{y^k, v^k, \mu^k\}$ be the sequence defined in Eqs. (22)-(25). Consider a sample path of Ψ^k and Φ^k along which the sequence $\left\{ \|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2 \right\}$ converges to 0 and the sequence $\{z^k, p^k\}$ is bounded, where r^k is the residual defined as in Eq. (27). Then, the sequence $\{x^k, y^k, z^k\}$ has a limit point, which is a saddle point of the Lagrangian function of problem (1).

B. Convergence and Rate of Convergence

The results of the previous section did not rely on the probability distributions of random variables Φ^k and Ψ^k . In this section, we will introduce a weighted norm and weighted Lagrangian function where the weights are defined in terms of the probability distributions of random variables Ψ^k and Φ^k representing the active constraints and components. We will use the weighted norm to construct a nonnegative supermartingale along the sequence $\{x^k, z^k, p^k\}$ generated by the asynchronous ADMM algorithm and use it to establish the almost sure convergence of this sequence to a saddle point of the Lagrangian function of problem (1). By relating the iterates generated by the asynchronous ADMM

algorithm to the variables (y^k, v^k, μ^k) through taking expectations of the weighted Lagrangian function and using results from Theorem 4.2, we will show that under a compactness assumption on the constraint sets X and Z , the asynchronous ADMM algorithm converges with rate $O(1/k)$ in expectation in terms of both objective function value and constraint violation.

We use the notation α_i to denote the probability that component x_i is active at one iteration, i.e.,

$$\alpha_i = \mathbb{P}(i \in \Phi^k), \quad (31)$$

and the notation λ_l to denote the probability that constraint l is active at one iteration, i.e.,

$$\lambda_l = \mathbb{P}(l \in \Psi^k). \quad (32)$$

Note that, since the random variables Φ^k (and Ψ^k) are independent and identically distributed for all k , these probabilities are the same across all iterations. We define a diagonal matrix Λ in $\mathbb{R}^{W \times W}$ with elements λ_l on the diagonal, i.e.,

$$\Lambda_{ll} = \lambda_l \quad \text{for each } l \in \{1, \dots, W\}.$$

Since each constraint is assumed to be active with strictly positive probability [cf. Assumption 4], matrix Λ is positive definite. We write $\bar{\Lambda}$ to indicate the inverse of matrix Λ . Matrix $\bar{\Lambda}$ induces a *weighted vector norm* for p in \mathbb{R}^W as

$$\|p\|_{\bar{\Lambda}}^2 = p' \bar{\Lambda} p.$$

We define a *weighted Lagrangian function* $\tilde{L}(x, z, \mu) : \mathbb{R}^{nN} \times \mathbb{R}^W \times \mathbb{R}^W \rightarrow \mathbb{R}$ as

$$\tilde{L}(x, z, \mu) = \sum_{i=1}^N \frac{1}{\alpha_i} f_i(x_i) - \mu' \left(\sum_{i=1}^N \frac{1}{\alpha_i} D_i x + \sum_{l=1}^W \frac{1}{\lambda_l} H_l z \right). \quad (33)$$

We use the symbol \mathcal{J}_k to denote the filtration up to and include iteration k , which contains information of random variables Φ^t and Ψ^t for $t \leq k$. We have $\mathcal{J}_k \subset \mathcal{J}_{k+1}$ for all $k \geq 1$.

The particular weights in $\bar{\Lambda}$ -norm and the weighted Lagrangian function are chosen to relate the expectation of the norm $\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - v)\|_{\bar{\Lambda}}^2$ and function $\tilde{L}(x^{k+1}, z^{k+1}, \mu)$ to $\frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - v)\|_{\bar{\Lambda}}^2$ and function $\tilde{L}(x^k, z^k, \mu)$, as we will show in the following lemma. This relation will be used in Theorem 4.6 to show that the scalar sequence $\left\{ \frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - v)\|_{\bar{\Lambda}}^2 \right\}$ is a nonnegative supermartingale, and establish almost sure convergence of the asynchronous ADMM algorithm.

Lemma 4.4: Let $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm (12)-

(14). Let $\{y^k, v^k, \mu^k\}$ be the sequence defined in Eqs. (22)-(25). Then the following hold for each iteration k :

$$\begin{aligned} \mathbb{E} \left(\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - v)\|_{\bar{\Lambda}}^2 \middle| \mathcal{J}_k \right) &= \frac{1}{2\beta} \|\mu^{k+1} - \mu\|^2 \\ &+ \frac{\beta}{2} \|H(v^{k+1} - v)\|^2 + \frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - v)\|_{\bar{\Lambda}}^2 - \frac{1}{2\beta} \|p^k - \mu\|^2 - \frac{\beta}{2} \|H(z^k - v)\|^2, \end{aligned} \quad (34)$$

for all μ in \mathbb{R}^W and v in Z , and

$$\begin{aligned} \mathbb{E} \left(\tilde{L}(x^{k+1}, z^{k+1}, \mu) \middle| \mathcal{J}_k \right) \\ = (F(y^{k+1}) - \mu'(Dy^{k+1} + Hv^{k+1})) + \tilde{L}(x^k, z^k, \mu) - (F(x^k) - \mu'(Dx^k + Hz^k)), \end{aligned} \quad (35)$$

for all μ in \mathbb{R}^W .

Proof: By the definition of λ_l in Eq. (32), for each l , the element p_l^{k+1} can be either updated to μ_l^{k+1} with probability λ_l , or stay at previous value p_l^k with probability $1 - \lambda_l$. Hence, we have the following expected value

$$\begin{aligned} \mathbb{E} \left(\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\bar{\Lambda}}^2 \middle| \mathcal{J}_k \right) &= \sum_{l=1}^W \frac{1}{\lambda_l} \left[\lambda_l \left(\frac{1}{2\beta} \|\mu_l^{k+1} - \mu\|^2 \right) + (1 - \lambda_l) \left(\frac{1}{2\beta} \|p_l^k - \mu\|^2 \right) \right] \\ &= \frac{1}{2\beta} \|\mu^{k+1} - \mu\|^2 + \frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 - \frac{1}{2\beta} \|p^k - \mu\|^2, \end{aligned}$$

where the second equality follows from definition of $\|\cdot\|_{\bar{\Lambda}}$, and grouping the terms.

Similarly, z_l^{k+1} is either equal to v_l^{k+1} with probability λ_l or z_l^k with probability $1 - \lambda_l$. Due to the diagonal structure of the H matrix, the vector $H_l z$ has only one non-zero element equal to $[Hz]_l$ at l^{th} position and zeros else where. Thus, we obtain

$$\begin{aligned} \mathbb{E} \left(\frac{\beta}{2} \|H(z^{k+1} - v)\|_{\bar{\Lambda}}^2 \middle| \mathcal{J}_k \right) &= \sum_{l=1}^W \frac{1}{\lambda_l} \left[\lambda_l \left(\frac{\beta}{2} \|H_l(v^{k+1} - v)\|^2 \right) + (1 - \lambda_l) \left(\frac{\beta}{2} \|H_l(z^k - v)\|^2 \right) \right] \\ &= \frac{\beta}{2} \|H(v^{k+1} - v)\|^2 + \frac{\beta}{2} \|H(z^k - v)\|_{\bar{\Lambda}}^2 - \frac{\beta}{2} \|H(z^k - v)\|^2, \end{aligned}$$

where we used the definition of $\|\cdot\|_{\bar{\Lambda}}$ once again. By summing the above two equations and using linearity of expectation operator, we obtain Eq. (34).

Using a similar line of argument, we observe that at iteration k , for each i , x_i^{k+1} has the value of y_i^{k+1} with probability α_i and its previous value x_i^k with probability $1 - \alpha_i$. The expectation of function

\tilde{L} therefore satisfies

$$\begin{aligned} \mathbb{E} \left(\tilde{L}(x^{k+1}, z^{k+1}, \mu) \middle| \mathcal{J}_k \right) &= \sum_{i=1}^N \frac{1}{\alpha_i} [\alpha_i (f_i(y_i^{k+1}) - \mu' D_i y^{k+1}) + (1 - \alpha_i) (f_i(x_i^k) - \mu' D_i x^k)] \\ &+ \sum_{l=1}^W \frac{1}{\lambda_l} \mu' [\lambda_l H_l v^{k+1} + (1 - \lambda_l) H_l z^k] \\ &= \left(\sum_{i=1}^N f_i(y_i^{k+1}) - \mu' (D y^{k+1} + H v^{k+1}) \right) + \tilde{L}(x^k, z^k, \mu) - \left(\sum_{i=1}^N f_i(x_i^k) - \mu' (D x^k + H z^k) \right), \end{aligned}$$

where we used the fact that $D = \sum_{i=1}^N D_i$. Using the definition $F(x) = \sum_{i=1}^N f_i(x)$ [cf. Eq. (9)], this shows Eq. (35). ■

The next lemma builds on Lemma 4.3 and establishes a sufficient condition for the sequence $\{x^k, z^k, p^k\}$ to converge to a saddle point of the Lagrangian. Theorem 4.6 will then show that this sufficient condition holds with probability 1 and thus the algorithm converges almost surely.

Lemma 4.5: Let (x^*, z^*, p^*) be any saddle point of the Lagrangian function of problem (1) and $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm (12)-(14). Along any sample path of Φ^k and Ψ^k , if the scalar sequence $\frac{1}{2\beta} \|p^{k+1} - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\bar{\Lambda}}^2$ is convergent and the scalar sequence $\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2]$ converges to 0, then the sequence $\{x^k, z^k, p^k\}$ converges to a saddle point of the Lagrangian function of problem (1).

Proof: Since the scalar sequence $\frac{1}{2\beta} \|p^{k+1} - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\bar{\Lambda}}^2$ converges, matrix $\bar{\Lambda}$ is positive definite, and matrix H is invertible [cf. Assumption 3], it follows that the sequences $\{p^k\}$ and $\{z^k\}$ are bounded. Lemma 4.3 then implies that the sequence $\{x^k, z^k, p^k\}$ has a limit point.

We next show that the sequence $\{x^k, z^k, p^k\}$ has a unique limit point. Let $(\tilde{x}, \tilde{z}, \tilde{p})$ be a limit point of the sequence $\{x^k, z^k, p^k\}$, i.e., the limit of sequence $\{x^k, z^k, p^k\}$ along a subsequence κ . We first show that the components \tilde{z}, \tilde{p} are uniquely defined. By Lemma 4.3, the point $(\tilde{x}, \tilde{z}, \tilde{p})$ is a saddle point of the Lagrangian function. Using the assumption of the lemma for $(p^*, z^*) = (\tilde{p}, \tilde{z})$, this shows that the scalar sequence $\left\{ \frac{1}{2\beta} \|p^{k+1} - \tilde{p}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - \tilde{z})\|_{\bar{\Lambda}}^2 \right\}$ is convergent. The limit of the sequence, therefore, is the same as the limit along any subsequence, implying

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{2\beta} \|p^{k+1} - \tilde{p}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - \tilde{z})\|_{\bar{\Lambda}}^2 &= \lim_{k \rightarrow \infty, k \in \kappa} \frac{1}{2\beta} \|p^{k+1} - \tilde{p}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - \tilde{z})\|_{\bar{\Lambda}}^2 \\ &= \frac{1}{2\beta} \|\tilde{p} - \tilde{p}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(\tilde{z} - \tilde{z})\|_{\bar{\Lambda}}^2 = 0, \end{aligned}$$

Since matrix $\bar{\Lambda}$ is positive definite and matrix H is invertible, this shows that $\lim_{k \rightarrow \infty} p^k = \tilde{p}$ and $\lim_{k \rightarrow \infty} z^k = \tilde{z}$.

Next we prove that given (\tilde{z}, \tilde{p}) , the x component of the saddle point is uniquely determined. By Lemma 4.1, we have $D\tilde{x} + H\tilde{z} = 0$. Since matrix D has full column rank [cf. Assumption 3], the vector \tilde{x} is uniquely determined by $\tilde{x} = -(D'D)^{-1}D'H\tilde{z}$. \blacksquare

The next theorem establishes almost sure convergence of the asynchronous ADMM algorithm. Our analysis uses results related to supermartingales (interested readers are referred to [18] and [48] for a comprehensive treatment of the subject).

Theorem 4.6: Let $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm (12)-(14). The sequence $\{x^k, z^k, p^k\}$ converges almost surely to a saddle point of the Lagrangian function of problem (1).

Proof: We will show that the conditions of Lemma 4.5 are satisfied almost surely. We will first focus on the scalar sequence $\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^{k+1} - v)\|_{\Lambda}^2$ and show that it is a nonnegative supermartingale. By martingale convergence theorem, this shows that it converges almost surely. We next establish that the scalar sequence $\frac{\beta}{2} \|r^{k+1} - H(v^{k+1} - z^k)\|^2$ converges to 0 almost surely by an argument similar to the one used to establish Borel-Cantelli lemma. These two results imply that the set of events where $\frac{1}{2\beta} \|p^{k+1} - p^*\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\Lambda}^2$ is convergent and $\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2]$ converges to 0 has probability 1. Hence, by Lemma 4.5, we have the sequence $\{x^k, z^k, p^k\}$ converges to a saddle point of the Lagrangian function almost surely.

We first show that the scalar sequence $\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^{k+1} - v)\|_{\Lambda}^2$ is a nonnegative supermartingale. Since it is a summation of two norms, it immediately follows that it is nonnegative. To see it is a supermartingale, we let vectors $y^{k+1}, v^{k+1}, \mu^{k+1}$ and r^{k+1} be those defined in Eqs. (22), (24), (25) and (27). Recall that the symbol \mathcal{J}_k denotes the filtration up to and including iteration k . From Lemma 4.4, we have

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{2\beta} \|p^{k+1} - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^{k+1} - v)\|_{\Lambda}^2 \middle| \mathcal{J}_k \right) \\ &= \frac{1}{2\beta} \|\mu^{k+1} - \mu\|^2 + \frac{\beta}{2} \|H(v^{k+1} - v)\|^2 + \frac{1}{2\beta} \|p^k - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^k - v)\|_{\Lambda}^2 \\ & \quad - \frac{1}{2\beta} \|p^k - \mu\|^2 - \frac{\beta}{2} \|H(z^k - v)\|^2 \end{aligned}$$

Substituting $\mu = p^*$ and $v = z^*$ in the above expectation calculation and combining with the following inequality from Theorem 4.2,

$$\begin{aligned} 0 & \geq \frac{1}{2\beta} \left(\|\mu^{k+1} - p^*\|^2 - \|p^k - p^*\|^2 \right) + \frac{\beta}{2} \left(\|H(v^{k+1} - z^*)\|^2 - \|H(z^k - z^*)\|^2 \right) \\ & \quad + \frac{\beta}{2} \|r^{k+1}\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2, \end{aligned}$$

we obtain

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{2\beta} \|p^{k+1} - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\bar{\Lambda}}^2 \middle| \mathcal{J}_k \right) \\ & \leq \frac{1}{2\beta} \|p^k - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - z^*)\|_{\bar{\Lambda}}^2 - \frac{\beta}{2} \|r^{k+1}\|^2 - \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2. \end{aligned}$$

Hence, the sequence $\frac{1}{2\beta} \|p^{k+1} - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\bar{\Lambda}}^2$ is a nonnegative supermartingale in k and by martingale convergence theorem, it converges almost surely.

We next establish that the scalar sequence $\left\{ \frac{\beta}{2} \|r^{k+1}\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2 \right\}$ converges to 0 almost surely. Rearranging the terms in the previous inequality and taking iterated expectation with respect to the filtration \mathcal{J}_k , we obtain for all T

$$\begin{aligned} \sum_{k=1}^T \mathbb{E} \left(\frac{\beta}{2} \|r^{k+1}\|^2 + \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2 \right) & \leq \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \\ & \quad - \mathbb{E} \left(\frac{1}{2\beta} \|p^{T+1} - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{T+1} - z^*)\|_{\bar{\Lambda}}^2 \right) \\ & \leq \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2, \end{aligned} \quad (36)$$

where the last inequality follows from relaxing the upper bound by dropping the non-positive expected value term. Thus, the sequence $\left\{ \mathbb{E} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \right) \right\}$ is summable implying

$$\lim_{k \rightarrow \infty} \sum_{t=k}^{\infty} \mathbb{E} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \right) = 0 \quad (37)$$

By Markov inequality, we have

$$\mathbb{P} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \geq \epsilon \right) \leq \frac{1}{\epsilon} \mathbb{E} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \right),$$

for any scalar $\epsilon > 0$ for all iterations t . Therefore, we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \mathbb{P} \left(\sup_{t \geq k} \frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \geq \epsilon \right) \\ & = \lim_{k \rightarrow \infty} \mathbb{P} \left(\bigcup_{t=k}^{\infty} \frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \geq \epsilon \right) \\ & \leq \lim_{k \rightarrow \infty} \sum_{t=k}^{\infty} \mathbb{P} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \geq \epsilon \right) \\ & \leq \lim_{k \rightarrow \infty} \frac{1}{\epsilon} \sum_{t=k}^{\infty} \mathbb{E} \left(\frac{\beta}{2} [\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2] \right) = 0, \end{aligned}$$

where the first inequality follows from union bound on probability, the second inequality follows from the preceding relation, and the last equality follows from Eq. (37). This proves that the sequence $\frac{\beta}{2} \left[\|r^{k+1}\|^2 + \|H(v^{k+1} - z^k)\|^2 \right]$ converges to 0 almost surely. ■

We next analyze convergence rate of the asynchronous ADMM algorithm. The rate analysis is done with respect to the time ergodic averages defined as $\bar{x}(T)$ in \mathbb{R}^{nN} , the time average of x^k up to and including iteration T , i.e.,

$$\bar{x}_i(T) = \frac{\sum_{k=1}^T x_i^k}{T}, \quad (38)$$

for all $i = 1, \dots, N$,¹² and $\bar{z}(k)$ in \mathbb{R}^W as

$$\bar{z}_l(T) = \frac{\sum_{k=1}^T z_l^k}{T}, \quad (39)$$

for all $l = 1, \dots, W$.

We next introduce some scalars $Q(\mu)$, \bar{Q} , $\bar{\theta}$ and \tilde{L}^0 , all of which will be used to provide an upper bound on the constant term that appears in the rate analysis. Scalar $Q(\mu)$ is defined by

$$Q(\mu) = \max_{x \in X, z \in Z} -\tilde{L}(x, z, \mu), \quad (40)$$

which implies $Q(\mu) \geq -\tilde{L}(x^{k+1}, z^{k+1}, \mu)$ for any realization of Ψ^k and Φ^k . For the rest of the section, we adopt the following assumption, which will be used to guarantee that scalar $Q(\mu)$ is well defined and finite:

Assumption 5: The sets X and Z are both compact.

Since the weighted Lagrangian function \tilde{L} is continuous in x and z [cf. Eq. (33)], and all iterates (x^k, z^k) are in the compact set $X \times Z$, by Weierstrass theorem the maximization in the preceding equality is attained and finite.

Since function \tilde{L} is linear in μ , the function $Q(\mu)$ is the maximum of linear functions and is thus convex and continuous in μ . We define scalar \bar{Q} as

$$\bar{Q} = \max_{\mu = p^* - \alpha, \|\alpha\| \leq 1} Q(\mu). \quad (41)$$

The reason that such scalar $\bar{Q} < \infty$ exists is once again by Weierstrass theorem (maximization over a compact set).

¹²Here the notation $\bar{x}_i(T)$ denotes the vector of length n corresponding to agent i .

We define vector $\bar{\theta}$ in \mathbb{R}^W as

$$\bar{\theta} = p^* - \operatorname{argmax}_{\|u\| \leq 1} \|p^0 - (p^* - u)\|_{\bar{\Lambda}}^2, \quad (42)$$

such maximizer exists due to Weierstrass theorem and the fact that the set $\|u\| \leq 1$ is compact and the function $\|p^0 - (p^* - u)\|_{\bar{\Lambda}}^2$ is continuous. Scalar \tilde{L}^0 is defined by

$$\tilde{L}^0 = \max_{\theta = p^* - \alpha, \|\alpha\| \leq 1} \tilde{L}(x^0, z^0, \theta). \quad (43)$$

This scalar is well defined because the constraint set is compact and the function \tilde{L} is continuous in θ .

Theorem 4.7: Let $\{x^k, z^k, p^k\}$ be the sequence generated by the asynchronous ADMM algorithm 12)-(14) and (x^*, z^*, p^*) be a saddle point of the Lagrangian function of problem (1). Let the vectors $\bar{x}(T), \bar{z}(T)$ be defined as in Eqs. (38) and (39), the scalars $\bar{Q}, \bar{\theta}$ and \tilde{L}^0 be defined as in Eqs. (41), (42) and (43) and the function \tilde{L} be defined as in Eq. (33). Then the following relations hold:

$$\|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\| \leq \frac{1}{T} \left[\bar{Q} + \tilde{L}^0 + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right], \quad (44)$$

and

$$\begin{aligned} \|\mathbb{E}(F(\bar{x}(T))) - F(x^*)\| &\leq \frac{1}{T} \left[\bar{Q} + \tilde{L}^0 + \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right] \\ &\quad + \frac{\|p^*\|_{\infty}}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right]. \end{aligned} \quad (45)$$

Proof: The proof of the theorem relies on Lemma 4.4 and Theorem 4.2. We combine these results with law of iterated expectation, telescoping cancellation and convexity of the function F to establish the bound

$$\begin{aligned} \mathbb{E} [F(\bar{x}(T)) - \mu'(D\bar{x}(T) + H\bar{z}(T))] - F(x^*) \\ \leq \frac{1}{T} \left[Q(\mu) + \tilde{L}(x^0, z^0, \mu) + \frac{1}{2\beta} \|p^0 - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right], \end{aligned} \quad (46)$$

for all μ in \mathbb{R}^W . Then by using different choices of the vector μ , we obtain the desired results.

We will first prove Eq. (46). Recall Eq. (35):

$$\begin{aligned} \mathbb{E} \left(\tilde{L}(x^{k+1}, z^{k+1}, \mu) \middle| \mathcal{J}_k \right) \\ = (F(y^{k+1}) - \mu'(Dy^{k+1} + Hv^{k+1})) + \tilde{L}(x^k, z^k, \mu) - (F(x^k) - \mu'(Dx^k + Hz^k)), \end{aligned}$$

We rearrange Eq. (29) from Theorem 4.2, and obtain

$$\begin{aligned} F(y^{k+1}) - \mu' r^{k+1} &\leq F(x^*) - \frac{1}{2\beta} \left(\|\mu^{k+1} - \mu\|^2 - \|p^k - \mu\|^2 \right) \\ &\quad - \frac{\beta}{2} \left(\|H(v^{k+1} - z^*)\|^2 - \|H(z^k - z^*)\|^2 \right) - \frac{\beta}{2} \|r^{k+1}\|^2 - \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2. \end{aligned}$$

Since $r^{k+1} = Dy^{k+1} + Hv^{k+1}$, we can apply this bound on the first term on the right-hand side of the preceding relation which implies

$$\begin{aligned} \mathbb{E} \left(\tilde{L}(x^{k+1}, z^{k+1}, \mu) \middle| \mathcal{J}_k \right) &\leq F(x^*) - \frac{1}{2\beta} \left(\|\mu^{k+1} - \mu\|^2 - \|p^k - \mu\|^2 \right) \\ &\quad - \frac{\beta}{2} \left(\|H(v^{k+1} - z^*)\|^2 - \|H(z^k - z^*)\|^2 \right) - \frac{\beta}{2} \|r^{k+1}\|^2 - \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2 \\ &\quad + \tilde{L}(x^k, z^k, \mu) - (F(x^k) - \mu'(Dx^k + Hz^k)), \end{aligned}$$

Combining the above inequality with Eq. (34) and using the linearity of expectation, we have

$$\begin{aligned} &\mathbb{E} \left(\tilde{L}(x^{k+1}, z^{k+1}, \mu) + \frac{1}{2\beta} \|p^{k+1} - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{k+1} - z^*)\|_{\bar{\Lambda}}^2 \middle| \mathcal{J}_k \right) \\ &\leq F(x^*) - \frac{\beta}{2} \|r^{k+1}\|^2 - \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2 - (F(x^k) - \mu'(Dx^k + Hz^k)) \\ &\quad + \tilde{L}(x^k, z^k, \mu) + \frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - z^*)\|_{\bar{\Lambda}}^2 \\ &\leq F(x^*) - (F(x^k) - \mu'(Dx^k + Hz^k)) + \tilde{L}(x^k, z^k, \mu) + \frac{1}{2\beta} \|p^k - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^k - z^*)\|_{\bar{\Lambda}}^2, \end{aligned}$$

where the last inequality follows from relaxing the upper bound by dropping the non-positive term $-\frac{\beta}{2} \|r^{k+1}\|^2 - \frac{\beta}{2} \|H(v^{k+1} - z^k)\|^2$.

This relation holds for $k = 1, \dots, T$ and by the law of iterated expectation, the telescoping sum after term cancellation satisfies

$$\begin{aligned} &\mathbb{E} \left(\tilde{L}(x^{T+1}, z^{T+1}, \mu) + \frac{1}{2\beta} \|p^{T+1} - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^{T+1} - z^*)\|_{\bar{\Lambda}}^2 \right) \leq TF(x^*) \tag{47} \\ &\quad - \mathbb{E} \left[\sum_{k=1}^T (F(x^k) - \mu'(Dx^k + Hz^k)) \right] + \tilde{L}(x^0, z^0, \mu) + \frac{1}{2\beta} \|p^0 - \mu\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2. \end{aligned}$$

By convexity of the functions f_i , we have

$$\sum_{k=1}^T F(x^k) = \sum_{k=1}^T \sum_{i=1}^N f_i(x_i^k) \geq T \sum_{i=1}^N f_i(\bar{x}_i(T)) = TF(\bar{x}(T)).$$

The same results hold after taking expectation on both sides. By linearity of matrix-vector multiplication,

we have $\sum_{k=1}^T Dx^k = TD\bar{x}(T)$, $\sum_{k=1}^T Hz^k = TH\bar{z}(T)$. Relation (47) therefore implies that

$$\begin{aligned} & T\mathbb{E}[F(\bar{x}(T)) - \mu'(D\bar{x}(T) + H\bar{z}(T))] - TF(x^*) \\ & \leq \mathbb{E}\left[\sum_{k=1}^T (F(x^k) - \mu'(Dx^k + Hz^k))\right] - TF(x^*) \\ & \leq -\mathbb{E}\left(\tilde{L}(x^{T+1}, z^{T+1}, \mu) + \frac{1}{2\beta} \|p^{T+1} - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^{T+1} - z^*)\|_{\Lambda}^2\right) \\ & \quad + \tilde{L}(x^0, z^0, \mu) + \frac{1}{2\beta} \|p^0 - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\Lambda}^2. \end{aligned}$$

Using the definition of scalar $Q(\mu)$ [cf. Eq. (40)] and by dropping the non-positive norm terms from the above upper bound, we obtain

$$\begin{aligned} & T\mathbb{E}[F(\bar{x}(T)) - \mu'(D\bar{x}(T) + H\bar{z}(T))] - TF(x^*) \\ & \leq Q(\mu) + \tilde{L}(x^0, z^0, \mu) + \frac{1}{2\beta} \|p^0 - \mu\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\Lambda}^2. \end{aligned}$$

We now divide both sides of the preceding inequality by T and obtain Eq. (46).

We now use Eq. (46) to first show that $\|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|$ converges to 0 with rate $1/T$. For each iteration T , we define a vector $\theta(T)$ as $\theta(T) = p^* - \frac{\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))}{\|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|}$. By substituting $\mu = \theta(T)$ in Eq. (46), we obtain for each T ,

$$\begin{aligned} & \mathbb{E}[F(\bar{x}(T)) - (\theta(T))'(D\bar{x}(T) + H\bar{z}(T))] - F(x^*) \\ & \leq \frac{1}{T} \left[Q(\theta(T)) + \tilde{L}(x^0, z^0, \theta(T)) + \frac{1}{2\beta} \|p^0 - \theta(T)\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\Lambda}^2 \right], \end{aligned}$$

Since the vectors $\frac{\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))}{\|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|}$ all have norm 1 and hence $\theta(T)$ is bounded within the unit sphere, by using the definition of $\bar{\theta}$, we have $\|p^0 - \theta(T)\|_{\Lambda}^2 \leq \|p^0 - \bar{\theta}\|_{\Lambda}^2$. Eqs. (41) and (43) implies $Q(\theta(T)) \leq \bar{Q}$ and $\tilde{L}(x^0, z^0, \theta(T)) \leq \tilde{L}^0$ for all T . Thus the above inequality suggests that the following holds true for all T ,

$$\mathbb{E}(F(\bar{x}(T)) - (\theta(T))'\mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) - F(x^*)) \leq \frac{1}{T} \left[\bar{Q} + \tilde{L}^0 + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\Lambda}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\Lambda}^2 \right].$$

From the definition of $\theta(T)$, we have $(\theta(T))'\mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) = (p^*)'\mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) - \|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|$, and thus

$$\begin{aligned} \mathbb{E}(F(\bar{x}(T)) - (\theta(T))'\mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) - F(x^*)) & = \mathbb{E}(F(\bar{x}(T))) - (p^*)'\mathbb{E}[(D\bar{x}(T) + H\bar{z}(T))] \\ & \quad - F(x^*) + \|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|. \end{aligned}$$

Since the point (x^*, z^*, p^*) is a saddle point of the Lagrangian function, using Lemma 4.1, we have

$$0 \leq \mathbb{E}L((\bar{x}(T)), \bar{z}(T), p^*) - L(x^*, z^*, p^*) = \mathbb{E}(F(\bar{x}(T))) - F(x^*) - (p^*)' \mathbb{E}[(D\bar{x}(T) + H\bar{z}(T))]. \quad (48)$$

The preceding three relations imply that

$$\|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\| \leq \frac{1}{T} \left[\bar{Q} + \tilde{L}^0 + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right],$$

which shows the first desired inequality.

To prove Eq. (45), we let $\mu = p^*$ in Eq. (46) and obtain

$$\begin{aligned} & \mathbb{E}(F(\bar{x}(T)) - (p^*)' \mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) - F(x^*)) \\ & \leq \frac{1}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right]. \end{aligned}$$

This inequality together with Eq. (48) imply

$$\begin{aligned} & \|\mathbb{E}(F(\bar{x}(T))) - (p^*)' \mathbb{E}(D\bar{x}(T) + H\bar{z}(T)) - F(x^*)\| \\ & \leq \frac{1}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right]. \end{aligned}$$

By triangle inequality, we obtain

$$\begin{aligned} \|\mathbb{E}(F(\bar{x}(T))) - F(x^*)\| & \leq \frac{1}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right] \\ & \quad + \|\mathbb{E}((p^*)'(D\bar{x}(T) + H\bar{z}(T)))\|, \end{aligned} \quad (49)$$

Using definition of Euclidean and l_∞ norms,¹³ the last term $\|\mathbb{E}((p^*)'(D\bar{x}(T) + H\bar{z}(T)))\|$ satisfies

$$\begin{aligned} \|\mathbb{E}((p^*)'(D\bar{x}(T) + H\bar{z}(T)))\| & = \sqrt{\sum_{l=1}^W (p_l^*)^2 [\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))]_l^2} \\ & \leq \sqrt{\sum_{l=1}^W \|p^*\|_\infty^2 [\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))]_l^2} = \|p^*\|_\infty \|\mathbb{E}(D\bar{x}(T) + H\bar{z}(T))\|. \end{aligned}$$

The above inequality combined with Eq. (44) yields

$$\|\mathbb{E}((p^*)'(D\bar{x}(T) + H\bar{z}(T)))\| \leq \frac{\|p^*\|_\infty}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right].$$

¹³We use the standard notation that $\|x\|_\infty = \max_i |x_i|$.

Hence, Eq. (49) implies

$$\begin{aligned} \|\mathbb{E}(F(\bar{x}(T))) - F(x^*)\| &\leq \frac{1}{T} \left[\bar{Q} + \tilde{L}^0 + \frac{1}{2\beta} \|p^0 - p^*\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right] \\ &\quad + \frac{\|p^*\|_{\infty}}{T} \left[Q(p^*) + \tilde{L}(x^0, z^0, p^*) + \frac{1}{2\beta} \|p^0 - \bar{\theta}\|_{\bar{\Lambda}}^2 + \frac{\beta}{2} \|H(z^0 - z^*)\|_{\bar{\Lambda}}^2 \right]. \end{aligned}$$

Thus we have established the desired relation (45). \blacksquare

We remark that by Jensen's inequality and convexity of the function F , we have $F(\mathbb{E}(\bar{x}(T))) \leq \mathbb{E}(F(\bar{x}(T)))$, and the preceding results also holds true when we replace $\mathbb{E}(F(\bar{x}(T)))$ by $F(\mathbb{E}(\bar{x}(T)))$.

V. CONCLUSIONS

We developed a fully asynchronous ADMM based algorithm for a convex optimization problem with separable objective function and linear constraints. This problem is motivated by distributed multi-agent optimization problems where a (static) network of agents each with access to a privately known local objective function seek to optimize the sum of these functions using computations based on local information and communication with neighbors. We show that this algorithm converges almost surely to an optimal solution. Moreover, the rate of convergence of the objective function values and feasibility violation is given by $O(1/k)$. Future work includes investigating network effects (e.g., effects of communication noise, quantization) and time-varying network topology on the performance of the algorithm.

REFERENCES

- [1] T. C. Aysal, M. E. Yildiz, A. D. Sarwate, and A. Scaglione. Broadcast gossip algorithms for consensus. *IEEE Transactions on Signal Processing*, 57(7):2748–2761, 2009.
- [2] D. P. Bertsekas and J. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, Belmont, MA, 1997.
- [3] C. Bishop. *Pattern Recognition and Machine Learning*. Springer, New York, 2006.
- [4] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. *Proceedings of IEEE Conference on Decision and Control (CDC)*, 2005.
- [5] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *Information Theory, IEEE Transactions on*, 52(6):2508–2530, 2006.
- [6] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. *Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers*, volume 3(1). Foundations and Trends in Machine Learning, 2010.
- [7] F. Bullo, J. Cortés, and S. Martinez. *Distributed Control of Robotic Networks: a Mathematical Approach to Motion Coordination Algorithms*. Princeton University Press, 2009.
- [8] W. Deng and W. Yin. On the Global and Linear Convergence of the Generalized Alternating Direction Method of Multipliers. *Rice University CAAM, Technical Report TR12-14*, 2012.
- [9] A. G. Dimakis, A. D. Sarwate, and M. J. Wainwright. Geographic Gossip: Efficient Aggregation for Sensor Networks. *Proceedings of International Conference on Information Processing in Sensor Networks*, pages 69–76, 2006.

- [10] J. Douglas and H. H. Rachford. On the Numerical Solution of Heat Conduction Problems in Two and Three Space Variables. *Transactions of the American Mathematical Society*, 82:421–439, 1956.
- [11] J. C Duchi, A. Agarwal, and M. J. Wainwright. Dual Averaging for Distributed Optimization : Convergence Analysis and Network Scaling. *IEEE Transactions on Automatic Control*, 57(3):592–606, 2012.
- [12] J. Eckstein. Augmented Lagrangian and Alternating Direction Methods for Convex Optimization : A Tutorial and Some Illustrative Computational Results Augmented Lagrangian and Alternating Direction Methods for Convex Optimization : A Tutorial and Some II. *Rutcor Research Report*, 2012.
- [13] J. Eckstein and D. P. Bertsekas. On the Douglas-Rachford Splitting Method and the Proximal Point Algorithm for Maximal Monotone Operators. *LIDS Report 1919*, 1989.
- [14] F. Fagnani and S. Zampieri. Randomized consensus algorithms over large scale networks. *IEEE Journal on Selected Areas in Communications*, 26(4):634–649, 2008.
- [15] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri. Average Consensus on Networks with Quantized Communication. *International Journal of Robust and Nonlinear Control*, 19(16):1787–1816, 2009.
- [16] D. Goldfarb and S. Ma. Fast Multiple Splitting Algorithms for Convex Optimization. *Under revision in SIAM Journal on Optimization*, 2010.
- [17] D. Goldfarb, S. Ma, and K. Scheinberg. Fast Alternating Linearization Methods for Minimizing the Sum of Two Convex Functions. *Mathematical Programming*, pages 1–34, 2012.
- [18] G. Grimmett and D. Stirzaker. *Probability and Random Processes*. Oxford Univeristy Press, Oxford, 2001.
- [19] D. Han and X. Yuan. A Note on the Alternating Direction Method of Multipliers. *Journal of Optimization Theory and Applications*, 155(1):227–238, 2012.
- [20] B. He and X. Yuan. On the $O(1/t)$ Convergence Rate of Alternating Direction Method. *Optimization Online*, 2011.
- [21] M. Hong and Z. Luo. On the Linear Convergence of the Alternating Direction Method of Multipliers. *Arxiv preprint*, 2012.
- [22] F. Iutzeler, P. Bianchi, P. Ciblat, and W. Hachem. Asynchronous Distributed Optimization using a Randomized Alternating Direction Method of Multipliers. *Submitted to IEEE Conference on Decision and Control (CDC)*, 2013.
- [23] A. Jadbabaie, J. Lin, and S. Morse. Coordination of Groups of Mobile Autonomous Agents using Nearest Neighbor Rules., *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.
- [24] D. Jakovetic, J. Xavier, and J. M. F. Moura. Cooperative Convex Optimization in Networked Systems: Augmented Lagrangian Algorithms with Directed Gossip Communication. *IEEE Transactions on Signal Processing*, 59(8):3889–3902, 2011.
- [25] D. Jakovetic, J. Xavier, and J. M. F. Moura. Fast Distributed Gradient Methods. *Arxiv preprint*, 2011.
- [26] B. Johansson, T. Keviczky, M. Johansson, and K. H. Johansson. Subgradient Methods and Consensus Algorithms for Solving Convex Optimization Problems. *Proceedings of IEEE Conference on Decision and Control(CDC)*, pages 4185–4190, 2008.
- [27] S. Kar and J. M. F. Moura. Distributed Consensus Algorithms in Sensor Networks With Imperfect Communication: Link Failures and Channel Noise. *IEEE Transactions on Signal Processing*, 57(1):355–369, 2009.
- [28] P. L. Lions and B. Mercier. Splitting Algorithms for the Sum of Two Nonlinear Operators. *SIAM Journal on Numerical Analysis*, 16:964–979, 1979.
- [29] I. Lobel and A. Ozdaglar. Convergence Analysis of Distributed Subgradient Methods over Random Networks. *Proceedings of Annual Allerton Conference on Communication, Control, and Computing*, 2008.
- [30] I. Lobel, A. Ozdaglar, and D. Feijer. Distributed Multi-agent Optimization with State-Dependent Communication. *Mathematical Programming, special issue in honor of Paul Tseng*, 129(2):255–284, 2011.
- [31] I. Matei and J. S. Baras. Performance Evaluation of the Consensus-Based Distributed Subgradient Method Under Random Communication Topologies. *IEEE Journal of Selected Topics in Signal Processing*, 5(4):754–771, 2011.
- [32] J. Mota, J Xavier, P. Aguiar, and M. Püschel. ADMM For Consensus On Colored Networks. *Proceedings of IEEE Conference on Decision and Control (CDC)*, 2012.

- [33] J. Mota, J. Xavier, P. Aguiar, and M. Püschel. D-ADMM : A Communication-Efficient Distributed Algorithm For Separable Optimization. *Submitted to IEEE Transactions on Signal Processing*, 2012.
- [34] A. Nedich, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis. Distributed Subgradient Algorithms and Quantization Effects. *Proceedings of IEEE Conference on Decision and Control (CDC)*, 2008.
- [35] A. Nedich and A. Ozdaglar. Distributed Subgradient Methods for Multi-agent Optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.
- [36] A. Nedich, A. Ozdaglar, and P. A. Parrilo. Constrained Consensus and Optimization in Multi-agent Networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.
- [37] Y. Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120(1):221–259, 2009.
- [38] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.
- [39] A. Olshevsky and J. N. Tsitsiklis. Convergence Speed in Distributed Consensus and Averaging. *SIAM Journal on Control and Optimization*, 48(1):33–35, 2009.
- [40] B. N. Oreshkin, M. J. Coates, and M. G. Rabbat. Optimization and Analysis of Distributed Averaging With Short Node Memory. *IEEE Transactions on Signal Processing*, 58(5):2850–2865, 2010.
- [41] S. S. Ram, A. Nedich, and V. V. Veeravalli. Asynchronous Gossip Algorithms for Stochastic Optimization. *Proceedings of IEEE Conference on Decision and Control (CDC)*, 2009.
- [42] S. S. Ram, A. Nedich, and V. V. Veeravalli. Distributed Stochastic Subgradient Projection Algorithms for Convex Optimization. *Journal of Optimization Theory and Applications*, 147(3):516–545, 2010.
- [43] I. D. Schizas, R. Ribeiro, and G. B. Giannakis. Consensus in Ad Hoc WSNs with Noisy Links - Part I: Distributed Estimation of Deterministic Signals. *IEEE Transactions on Singal Processing*, 56:350–364, 2008.
- [44] A. Tahbaz-Salehi and A. Jadbabaie. A Necessary and Sufficient Condition for Consensus Over Random Networks. *IEEE Transactions on Automatic Control*, 53(3):791–795, 2008.
- [45] J. N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation*. PhD thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 1984.
- [46] E. Wei and A. Ozdaglar. Distributed Alternating Direction Method of Multipliers. *Proceedings of IEEE Conference on Decision and Control (CDC)*, 2012.
- [47] E. Wei and A. Ozdaglar. On the $O(1/k)$ Convergence of Asynchronous Distributed Alternating Direction Method of Multipliers. *LIDS report 2906*, 2013.
- [48] D. Williams. *Probability with Martingales*. Cambridge University Press, Cambridge, 1991.
- [49] H. Zhu, A. Cano, and G. B. Giannakis. In-Network Channel Decoding Using Consensus on Log-Likelihood Ratio Averages. *Proceedings of Conference on Information Sciences and Systems (CISS)*, 2008.