Uniqueness of Generalized Equilibrium for Box Constrained Problems and Applications

Alp Simsek
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

Asuman E. Ozdaglar
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

Daron Acemoglu
Department of Economics
Massachusetts Institute of Technology

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Abstract

In recent work, Simsek-Ozdaglar-Acemoglu [5] prove a generalization of the Poincare-Hopf Theorem and establish sufficient conditions for the uniqueness of generalized critical points (generalized equilibria) under some regularity assumptions. In this paper, we restrict ourselves to functions on box-constrained regions and establish the uniqueness of the generalized critical point with weaker regularity assumptions than in [5]. We use our uniqueness result to show the uniqueness of equilibrium in two recent network control models.

1 Introduction

Recent models of network control in both wireline and wireless networks lead to nonconvex optimization formulations mainly due to two reasons:

(a) Transmission medium characteristics result in nonlinear dependencies between control variables,

(b) Presence of multiple heterogeneous agents or interactions between multiple same/cross layer protocols cannot be modeled as constrained optimization problems, but rather as typically nonconvex equilibrium problems.

In recent work [5], we have used topological tools to analyze stationary points of nonconvex optimization and equilibrium problems. In particular, we proved a generalization of the Poincare-Hopf Theorem and established sufficient conditions for the uniqueness of generalized critical points under some regularity assumptions. In this paper, we present further generalization and applications of our results in two recent network control models. We restrict ourselves to "box-constrained " regions and establish uniqueness of the generalized critical point under weaker regularity assumptions. As will be seen in Sections

3 and 4, relaxing the regularity assumptions allows us to avoid unnecessary restrictive assumptions in establishing uniqueness of equilibrium and stationary points.

Regarding notation, for a given matrix A, A^{ij} denotes its entry in i^{th} row and j^{th} column. For an $n \times n$ matrix A and $J \subset \{1, 2, .., n\}$, let $A|_J$ denote the principal submatrix of A which contains precisely the entries A^{ij} where $i, j \in J$. When X is a finite set, we use $|X|$ to denote its cardinality. If a function f is differentiable at x, then $\nabla f(x)$ denotes the gradient of f. If f is twice differentiable at x, then $H_f(x)$ denotes the Hessian of f at x. We say that f is continuously differentiable at x if f is continuously differentiable in an open set containing x.

Let $X_i^{\min}, X_i^{\max} \in \mathbb{R}$ be constants such that $X_i^{\min} < X_i^{\max}$ for all $i \in \{1, 2, .., n\}$. Let $M \subset \mathbb{R}^n$ be the compact region given by

$$M = \{x \in \mathbb{R}^n \mid X_i^{\min} \leq x \leq X_i^{\max},\ \forall\, i \in \{1, 2, .., n\}\}. \tag{1}$$

We refer to such a region as a *box-constrained* region. Given $x \in M$, denote the set of components corresponding to the non-binding constraints at x with

$$I^{NB}(x) = \{i \in \{1, 2, .., n\} \mid X_i^{\min} < x_i < X_i^{\max}\}.$$

Let U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a function. Let $\mathrm{Cr}(F, M)$ denote the set of *generalized critical points* of F over M (cf. [5]). For the box constrained regions, a vector $x \in \mathrm{Cr}(F, M)$ iff

$$\begin{aligned} F_i(x) &= 0,\ \forall\, i \in I^{NB}(x) && (2)\\ F_i(x) &\geq 0,\ \forall\, i \text{ such that } x_i = X_i^{\min} && (3)\\ F_i(x) &\leq 0,\ \forall\, i \text{ such that } x_i = X_i^{\max}. && (4) \end{aligned}$$

We say that $x \in M$ is a *complementary critical point* if the inequalities in (3) and (4) are strict. When F is differentiable at x, we let

$$\Gamma(x) = \nabla F(x)|_{I^{NB}(x)}.$$

We say that $x \in \mathrm{Cr}(F, M)$ is a *non-degenerate critical point* if F is continuously differentiable at x and $\Gamma(x)$ is non-singular.

The main result in Simsek-Acemoglu-Ozdaglar [5] corresponds to the following theorem for the case when M is a box constrained region.

Theorem 1 Let M be a region given by (1). Let U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Assume that every $x \in \mathrm{Cr}(F, M)$ is complementary and non-degenerate. Then, $\mathrm{Cr}(F, M)$ has finitely many elements and

$$\sum_{x \in \mathrm{Cr}(F,M)} \mathrm{sign}(\det(\Gamma(x))) = 1.$$

2 A Uniqueness Result without the Complementarity Condition

We introduce a stronger non-degeneracy condition to replace the complementarity condition. We first define the notion of a P-matrix.

Definition 1 An $n \times n$ matrix A is a *P-matrix* if all the determinants of its principal sub-matrices are positive, i.e. if $\det(A|_J) > 0$ for all $J \subset \{1, 2, .., n\}$.

Let

$$I^F(x) = \{i \in \{1, 2, .., n\} \mid F_i(x) = 0\}.$$

Note that, if $x \in \mathrm{Cr}(F, M)$, $I^N(x) \subset I^F(x)$ and if x is a complementary critical point, $I^N(x) = I^F(x)$. We say that $x \in \mathrm{Cr}(F, M)$ is a *P-critical point* if F is continuously differentiable at x and $\nabla F(x)|_{I^F(x)}$ is a P-matrix. Note that every P-critical point is non-degenerate but not vice versa. We have the following theorem which establishes the uniqueness of the generalized critical point when the generalized critical point is not necessarily complementary.

Theorem 2 Let M be a region given by (1). Let U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that every $x \in \mathrm{Cr}(F, M)$ is a P-critical point. Then, F has a unique generalized critical point over M.

For the proof, we need some preliminary results regarding the properties of P-matrices. The proof of the following lemma could be found in Facchinei-Pang [1], Chapter 3.5.

Lemma 1 Let K be a box constrained region in $\mathbb{R}^n$, U be an open set containing K, and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that the Jacobian $\nabla F(u)$ is a P-matrix for all $u \in K$. Then, for each $x, z \in K$ such that $x \neq z$, there exists a component $j \in \{1, 2, .., n\}$ such that

$$(F_j(x) - F_j(z))(x_j - z_j) > 0.$$

We need the following lemma which is slightly stronger than Lemma 1.

Lemma 2 Let K be a box constrained region in $\mathbb{R}^n$, U be an open set containing K, and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Let $J \subset \{1, 2, .., n\}$ be an index set and assume that $\nabla F(u)|_J$ is a P-matrix for all $u \in K$. Then, for each $x, z \in K$ such that $x \neq z$ and $x_i = z_i$ for all $i \notin J$, there exists a component $j \in J$ such that

$$(F_j(x) - F_j(z))(x_j - z_j) > 0.$$

Proof. Without loss of generality, assume that $J = \{1, 2, .., m\}$ for some $m \leq n$. Given $A \subset \mathbb{R}^n$, let

$$A|_J = \{u \in \mathbb{R}^m \ |(u_1, .., u_m, x_{m+1}, .., x_n) \in A\}.$$

Consider the continuously differentiable function $g : U|_J \mapsto \mathbb{R}^m$ given by

$$G_i(u) = F_i(u_1, .., u_m, x_{m+1}, .., x_n)$$

for all $i \in J$ and $u \in U|_J$. Since $z_i = x_i$, for all $i \notin J$, we have

$$(x_1, .., x_m) \in K|_J,\ (z_1, .., z_m) \in K|_J,\ K|J \subset U|_J$$

and

$$G_k(x_1, .., x_m) = F_k(x),\ G_k(z_1, .., z_m) = F_k(z),\ \text{for all } k \in J. \tag{5}$$

Moreover, it can be seen that

$$\nabla G(u) = \nabla F(u_1, .., u_m, x_1, .., x_{m+1})|_J,$$

which implies that $\nabla G(u)$ is a P-matrix for all $u \in K|_J$. Since $K|_J$ is a box constraint region in $\mathbb{R}^m$, applying Lemma 1 for the function G and the vectors $(x_1, .., x_m), (z_1, .., z_m) \in K|_J$, there exists $j \in \{1, 2, .., m\}$ such that

$$(x_j - z_j)\,(G_j(x_1, .., x_m) - G_j(z_1, .., z_m)) > 0.$$

Then, by Eq. (5), we have $(x_j - z_j)(F_j(x) - F_j(z)) > 0$ as desired.

We introduce the notion of an irregular pair to prove Theorem 2.

Definition 2 Let M be a region given by (1). Let U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. We say that (i, x) is an *irregular pair* if $x \in \mathrm{Cr}(F, M)$, $i \in I^F(x) - I^N(x)$, i.e. the inequality corresponding to i in either Eq. (3) or Eq. (4) is not strict. We denote the set of irregular pairs of F over M by

$$\mathcal{A}(F, M) = \{(i, x) \mid x \in \mathrm{Cr}(F, M),\ i \in I^F(x) - I^N(x)\}.$$

We note that every $x \in \mathrm{Cr}(F, M)$ is a complementary critical point if $\mathcal{A}(F, M) = \emptyset$. We need the following lemma, which shows that we can recursively remove the irregular pairs.

Lemma 3 Let M be a region given by (1). Let U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Assume that $x \in \mathrm{Cr}(F, M)$ is a P-critical point. Then,
(i) There exists an open set U_x containing x such that $U_x \cap \mathrm{Cr}(F, M) = \{x\}$, i.e. x is an isolated critical point.
(ii) Assume that $(k, x) \in \mathcal{A}(F, M)$. Then, there exists a function $\tilde{F} : U \mapsto \mathbb{R}^n$ such that
(a) $\mathrm{Cr}(\tilde{F}, M) = \mathrm{Cr}(F, M)$.
(b) $\mathcal{A}(\tilde{F}, M) = \mathcal{A}(F, M) - \{(k, x)\}$.
(c) $x \in \mathrm{Cr}(\tilde{F}, M)$ is a P-critical point.

Proof. Since F is continuously differentiable at x, there exists an open set $S_x \subset U$ containing x over which F is continuously differentiable. The determinant of each principal sub-matrix of $\nabla F|_{I^F(x)}$ can be viewed as a continuous function over S_x. Since there are finitely many such functions each of which is positive at x, there exists an open set $V_x \subset S_x$ containing x such that the determinant of each sub-matrix of $\nabla F(u)|_{I^F(x)}$ is positive for all $u \in V_x$, i.e. $\nabla F(u)|_{I^F(x)}$ is a P-matrix for all $u \in V_x$. Let U_x be an open box constrained region which is a subset of V_x and which is so small that $F_i(x) > 0$, [resp. $F_i(x) < 0$] [resp. $X_j^{\min} < x_j < X_j^{\max}$] implies $F_i(u) > 0$ [resp. $F_i(u) < 0$] [resp. $X_j^{\min} < u_j < X_j^{\max}$] for all $u \in U_x$. We will show that U_x satisfies the claims of the lemma.

(i) Assume, to get a contradiction, that $y \in U_x$ is a critical point of F over M such that $y \neq x$. Let $J = \{i \mid y_i \neq x_i\}$ and consider $i \in J$. We first claim that $i \in I^{NB}(y)$. If $i \in I^{NB}(x)$, by choice of U_x, $i \in I^{NB}(y)$. Else if $x_i = X_i^{\min}$ or $x_i = X_i^{\max}$, since $y_i \neq x_i$ and $y \in M$, we have $X_i^{\min} < y_i < X_i^{\max}$, i.e. $i \in I^{NB}(y)$, showing the claim. Since $y \in \mathrm{Cr}(F, M)$, we further have $i \in I^F(y)$. Then, by choice of U_x, we also have $i \in I^F(x)$. Thus, we have shown

$$J \subset I^F(x), \text{ and } J \subset I^F(y). \tag{6}$$

Then, for all $u \in U_x$, $\nabla F(u)|_J$ is a principal sub-matrix of $\nabla F(u)|_{I^F(x)}$ and hence is a P-matrix. Since $x_i = y_i$ for all $i \notin J$, by Lemma 2, there exists $j \in J$ such that

$$(x_j - y_j)(F_j(x) - F_j(y)) > 0,$$

which yields a contradiction in view of Eq. (6).

(ii) Since $k \notin I^N(x)$, we have either $x_k = X_k^{\min}$ or $x_k = X_k^{\max}$. Assume $x_k = X_k^{\min}$. Let $w : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable weight function such that

$$\begin{cases} w(x) = 1, \\ w(y) \geq 0, \text{ if } y \in U_x \\ w(y) = 0, \text{ if } y \notin U_x. \end{cases}$$

Let $\tilde{F} : U \mapsto \mathbb{R}^n$ be given by

$$\begin{cases} \tilde{F}_k(x) = F_k(x) + w(x), \text{ for all } x \in U, \\ \tilde{F}_i(x) = F_i(x), \text{ for all } i \neq k \text{ and } x \in U. \end{cases} \tag{7}$$

We will show that the function $\tilde{F}$ satisfies the claims of the lemma. We have,

$$\tilde{F}_k(x) > 0, \text{ and } \tilde{F}_i(x) = F_i(x) \text{ for } i \neq k. \tag{8}$$

Since $x \in \mathrm{Cr}(F, M)$, $\tilde{F}(x)$ satisfies (2), (3), (4) and thus $x \in \mathrm{Cr}(\tilde{F}, M)$. We have

$$I^{\tilde{F}}(x) = I^F(x) - \{k\}. \tag{9}$$

By Eq. (9) and the definition in (7), we have

$$\nabla \tilde{F}(x)|_{I^{\tilde{F}}(x)} = \nabla F(x)|_{I^{\tilde{F}}(x)}. \tag{10}$$

Since x is a P-critical point, $\nabla F(x)|_{I^F(x)}$ is a P-matrix. Then, by equations (9) and (10), $\nabla \tilde{F}(x)|_{I^{\tilde{F}}(x)}$ is a P-matrix. This shows that $x \in \mathrm{Cr}(\tilde{F}, M)$ is a P-critical point, completing the proof of part (ii)-(c) of the lemma.

We next claim that $\mathrm{Cr}(\tilde{F}, M) \cap U_x = \{x\}$. Assume that there exists a critical point y of $\tilde{F}$ in U_x such that $y \neq x$. Let $J = \{i \mid y_i \neq x_i\}$. As shown in the proof of part (i),

$$J \subset I^{\tilde{F}}(x) \subset I^F(x), \text{ and } J \subset I^{\tilde{F}}(y). \tag{11}$$

Then, for $u \in U_x$, $\nabla F(u)|_J$ is a principal sub-matrix of $\nabla F(u)|_{I^F(x)}$ and hence is a

P-matrix. Since $x_i = y_i$ for all $i \notin J$, by Lemma 2, there exists $j \in J$ such that

$$(y_j - x_j)(F_j(y) - F_j(x)) > 0. \tag{12}$$

If $j \neq k$, then, using Eq. (8) and Eq. (11),

$$F_j(y) = \tilde{F}_j(y) = 0, \text{ and } F_j(x) = 0,$$

which yields a contradiction. Else if $j = k$, then we have

$$0 = (y - x)_j \tilde{F}_j(y) = (y - x)_j F_j(y) + (y - x)_j w(y).$$

We have $(y - x)_j > 0$ since $x_k = X_k^{\min}$, and $w(y) \geq 0$ by definition. Then the preceding equation implies

$$(y - x)_j F_j(y) \leq 0, \tag{13}$$

which, since $F_j(x) = 0$, yields a contradiction in view of Eq. (12). This completes the proof of part (ii)-(a) of the lemma.

Since $\tilde{F}_k(x) = w(x) > 0$, $(k, x) \notin \mathcal{A}(F, M)$. Let $(i, x) \in \mathcal{A}(F, M)$ for some $i \neq k$. Then, since $\tilde{F}_i = F_i$, we have $(i, x) \in \mathcal{A}(\tilde{F}, M)$. Let $(i, y) \in \mathcal{A}(F, M)$ for some $y \in \mathrm{Cr}(F, M)$ such that $y \neq x$. Then by part (i) of this Lemma, $y \notin U_x$ and thus $F(y) = \tilde{F}(y)$. This implies, $(i, y) \in \mathcal{A}(\tilde{F}, M)$. We conclude that $\mathcal{A}(\tilde{F}, M) = \mathcal{A}(F, M) - \{k, x\}$, completing the proof of part (ii)-(b) of the lemma.

The proof for the case when $k \in I^{\max}(x)$ can be analogously given with the continuously differentiable weight function chosen such that

$$\begin{cases} w(x) = -1, \\ w(y) \leq 0, \text{ if } y \in U_x \\ w(y) = 0, \text{ if } y \notin U_x. \end{cases}$$

Q.E.D.

We are now ready to prove Theorem 2.

Proof of Theorem 2. We first claim that $\mathrm{Cr}(F, M)$ is a compact set. Let

$$A_i^1 = \{x \in M \mid F_i(x) = 0\}, \qquad \begin{aligned} A_i^2 &= \{x \in M \mid F_i(x) \geq 0 \text{ and } x_i = X_i^{\min}\}, \\ A_i^3 &= \{x \in M \mid F_i(x) \leq 0 \text{ and } x_i = X_i^{\max}\}. \end{aligned}$$

Since each of A_i^1, A_i^2, A_i^3 is compact, so is $A_i = A_i^1 \cup A_i^2 \cup A_i^3$. By equations (2), (3), and (4), we have

$$\mathrm{Cr}(F, M) = \bigcap_{i \in \{1,2,..,n\}} A_i.$$

Then, being the intersection of compact sets, $\mathrm{Cr}(F, M)$ is compact. We next claim that $\mathrm{Cr}(F, M)$ has a finite number of elements. By part (i) of Lemma 3, for each $x \in \mathrm{Cr}(F, M)$, there exists an open set U_x containing x such that $U_x \cap \mathrm{Cr}(F, M) = \{x\}$. Then, $\{U_x \mid x \in \mathrm{Cr}(F, M)\}$ is an open covering of the compact set $\mathrm{Cr}(F, M)$, which implies that it has a finite sub-covering. This further implies that $\mathrm{Cr}(F, M)$ has a finite number of elements.

We finally claim that there exists a function $G : U \mapsto \mathbb{R}^n$ such that $\mathrm{Cr}(G, M) = \mathrm{Cr}(F, M)$ and every $x \in \mathrm{Cr}(G, M)$ is complementary and non-degenerate. Let $F^0 = F$

and, for any $j \geq 0$ such that $\mathcal{A}(F^j, M) \neq \emptyset$, define

$$F^{j+1} = \tilde{F^j}$$

where $\tilde{F^j}$ is the modified function which satisfies the claim of part (ii) of Lemma 3 for the function F^j and an arbitrary $(k, x) \in \mathcal{A}(F^j, M)$. By part (ii) of Lemma 3,

$$|\mathcal{A}(F^{j+1}, M)| = |\mathcal{A}(F^j, M)| - 1.$$

Since $\mathrm{Cr}(F, M)$ has finitely many elements, $\mathcal{A}(F, M)$ has finitely many elements, which implies that there exists an integer $m \geq 0$ such that $\mathcal{A}(F^m, M) = \emptyset$. We let, $G = F^m$. Since $\mathcal{A}(G, M) = \emptyset$, every $x \in \mathrm{Cr}(G, M)$ is complementary and by part (iii) of Lemma 3, every $x \in \mathrm{Cr}(F, M)$ is a P-critical point, which implies that it is also non-degenerate. Thus, G satisfies the claim. Since, every $x \in \mathrm{Cr}(G, M)$ is complementary and non-degenerate, Theorem 1 applies to G and we have

$$\sum_{x \in \mathrm{Cr}(F,M)} \mathrm{sign}\left(\det\left(\nabla G(x)|_{I^G(x)}\right)\right) = 1. \tag{14}$$

For $x \in \mathrm{Cr}(G, M)$, we have $\det(\nabla G(x)|_{I^G(x)}) > 0$ since x is a P-critical point of G. Then, $\mathrm{Cr}(G, M)$ has a unique element in view of (14). Since $\mathrm{Cr}(F, M) = \mathrm{Cr}(G, M)$, we conclude that F has a unique generalized critical point over M as desired. **Q.E.D.**

We have the following corollary to Theorem 2 regarding non-convex optimization.

Corollary 1 Let M be a region given by (1). Let U be an open set containing M, and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Let $\mathrm{KKT}(f, M)$ denote the Karush-Kuhn-Tucker stationary points of f over the region M (cf. [5]) and assume that for every $x \in \mathrm{KKT}(f, M)$, $H_f(x)|_{I^{\nabla f}(x)}$ is a P-matrix. Then, $\mathrm{KKT}(f, M)$ has a unique element which is also the unique local (global) minimum of f over M.

Proof. It was established in Simsek-Ozdaglar-Acemoglu [5] that

$$x \in \mathrm{KKT}(f, M) \iff x \in \mathrm{Cr}(\nabla F, M).$$

Moreover, since $\nabla(\nabla f) = H_f$, the assumption of the corollary is equivalent to the fact that every $x \in \mathrm{Cr}(\nabla f, M)$ is a P-critical point. Then, by Theorem 2, $\mathrm{Cr}(\nabla f, M)$ has a unique element, which further implies that $\mathrm{KKT}(f, M)$ has a unique element. Since M is compact, f has a global (local) minimum of f over M. Since every local minimum of f is a KKT point from first order conditions, we conclude that the unique element in $\mathrm{KKT}(f, M)$ is also the unique local (global) minimum of f over M. **Q.E.D.**

3 Uniqueness for the Wireless Control Problem

One way to mitigate interference in a wireless network is to control the nodes' transmit powers. In an ad hoc wireless network, due to the lack of a central infrastructure, it is essential to develop distributed algorithms for power control. A distributed algorithm with provable convergence properties can be developed (by gradient descent methods) if the optimum for the power control problem can be equivalently characterized by the first order optimality conditions (as would be the case for a convex optimization problem).

The power optimization problem, however, is nonconvex due to the nature of the wireless interference. An alternative approach is to show the uniqueness of stationary points, which together with the existence of an optimal solution, would guarantee the sufficiency of first order optimality conditions. In recent work, Huang-Berry-Honig [2] establish the uniqueness of stationary points for the power optimization problem by transforming the problem to a convex problem. They consequently develop a distributed algorithm which converges to the unique optimum. In this section, we use Corollary 1 to provide an alternative method for establishing the uniqueness of stationary points for the power optimization problem.

Let $L = \{1, 2, ..., n\}$ denote the set of nodes and

$$\mathcal{P} = \prod_{a \in L} [P_i^{\min}, P_i^{\max}] \subset \mathbb{R}^n$$

denote the set of power vectors p such that each node $i \in L$ transmits at a power level p_i. Assume that $0 < P_i^{\min} < P_i^{\max}$ for each $i \in L$. For each node i, define the *received SINR (signal to noise ratio)* to be the function $\gamma : \mathcal{P} \mapsto \mathbb{R}$

$$\gamma_i(p) = \frac{p_i h_{ii}}{n_0 + \sum_{j \neq i} p_j h_{ij}} \tag{15}$$

where n_0, h_{ji} are positive constants. Assume that each node i has an increasing and strictly concave utility function $u_i : \mathbb{R} \mapsto \mathbb{R}$, which is a function of the received SINR. Let $f : M \mapsto \mathbb{R}$ be given by

$$f(p) = -\sum_{1 \leq i \leq n} u_i(\gamma_i(p)) \tag{16}$$

Then, the *wireless power control problem* is to find a power vector p which solves the following optimization problem.

$$\min_{p \in \mathcal{P}} f(p). \tag{17}$$

In general, the function f given by (15)-(16) is non-convex. Therefore, traditional strict convexity arguments cannot be used to establish the uniqueness of solution for the problem given by (17).

Proposition 1 Let

$$\gamma_i^{\min} = \min_{p \in \mathcal{P}} \gamma_i(p), \qquad \gamma_i^{\max} = \max_{p \in \mathcal{P}} \gamma_i(p),$$

and assume that each utility function satisfies the following assumption regarding its *coefficient of relative risk aversion.*

(A) $-\frac{\gamma_i u_i''(\gamma_i)}{u_i'(\gamma_i)} \in [1, 2], \qquad \forall\ \gamma_i \in [\gamma_i^{\min}, \gamma_i^{\max}]$.

Then, the function f given by (15)-(16) has a unique KKT stationary point and the problem given by (17) has a unique solution.

Proof. Let $A_{ij} = (h_{ij} p_j) \,/\, (h_{ii} p_i)$. Then, we have

$$\sum_{j \neq i} \gamma_i(p) A_{ij} < 1 \tag{18}$$

It can be seen that

$$
\begin{aligned}
\nabla f_i(p) &= -\frac{1}{p_i}\left(u_i'\gamma_i - \sum_{j\neq i} A_{ij}u_j'\gamma_j^2\right),\\
H_f^{ii}(p) &= -\frac{1}{p_i^2}\left(u_i''\gamma_i^2 + \sum_{j\neq i}\gamma_j^2 A_{ij}^2\left(u_j''\gamma_j^2 + 2u_j'\gamma_j\right)\right),\ \forall i \in \{1,2,..,n\},
\end{aligned}
$$

and

$$
H_f^{il}(p) = \frac{1}{p_i p_l}\left(A_{li}\gamma_i\left(u_i''\gamma_i^2 + u_i'\gamma_i\right) + \gamma_l A_{il}\left(u_l''\gamma_l^2 + u_l'\gamma_l\right) - \sum_{j\neq i,l}\gamma_j^2 A_{lj}A_{ij}\left(u_j''\gamma_j^2 + 2u_j'\gamma_j\right)\right),
$$

for all $i, l \in \{1,2,..,n\}$ such that $i \neq k$.

We claim that $H_f(p)|_{I^{\nabla f}(p)}$ is a P-matrix for all $p \in M$. Let

$$C = P\nabla F(p)P$$

where P is the $n \times n$ diagonal matrix with entries p_i in the diagonal, and note that $\nabla F(p)|_{I^{\nabla f}(p)}$ is a P-matrix if and only if $C|_{I^{\nabla f}(p)}$ is a P-matrix. We claim that $C|_{I^{\nabla f}(p)}$ is positive row diagonally dominant. Since u_i satisfies (A), we have, for all $i \in L$,

$$u_i''\gamma_i^2 + u_i'\gamma_i \leq 0 \tag{19}$$

and

$$u_i''\gamma_i^2 + 2u_i'\gamma_i \geq 0. \tag{20}$$

Then, for $i \in I^{\nabla f}(p)$, using $\nabla f_i(p) = 0$, we have

$$
\begin{aligned}
C^{ii}(p) - \sum_{l\neq i} C^{il}(p) &= (-u_i''\gamma_i^2 - u_i'\gamma_i)(1 - \sum_{l\neq i} A_{li}\gamma_i) - \sum_{j\neq i}\gamma_j A_{ij}(\sum_{l\neq j}\gamma_j A_{lj})\left(u_j''\gamma_j^2 + 2u_j'\gamma_j\right)\\
&\quad + \sum_{j\neq i}\gamma_j A_{ij}\left(u_j''\gamma_j^2 + 2u_j'\gamma_j\right)\\
&= (-u_i''\gamma_i^2 - u_i'\gamma_i)(1 - \sum_{l\neq i} A_{li}\gamma_i) + \sum_{j\neq i}\gamma_j A_{ij}(1 - \sum_{l\neq j} A_{lj}\gamma_j)\left(u_j''\gamma_j^2 + 2u_j'\gamma_j\right)\\
&> 0 \qquad (21)
\end{aligned}
$$

where the inequality follows from Equations (18), (19), and (20). The inequality is strict since since either Eq. (19) or Eq. (20) must be strict. Also by Equations (19) and (20), $C^{il}(p) < 0$ for all $i \neq l$. Then, Eq. (21) implies, for all $i \in I^{\nabla f}(p)$,

$$C^{ii}(p) - \sum_{l\neq i, l\in I^{\nabla f}(p)}\left|C^{il}(p)\right| > 0$$

showing that $C(p)|_{I^{\nabla f}(p)}$ is positive row diagonally dominant. Therefore, $C(p)|_{I^{\nabla f}(p)}$, and thus $H_f(p)|_{I^{\nabla f}(p)}$, is a P-matrix for all $p \in M$. This implies, in particular, $H_f(p)|_{I^{\nabla f}(p)}$ is a P-matrix for all $p \in \mathrm{KKT}(f, M)$, and by Corollary 1, f has a unique KKT stationary point and the wireless optimization problem given by (17) has a unique solution. **Q.E.D.**

4 Uniqueness for a Heterogenous Network Protocol Model

In this section, we investigate the network model by Tang-Wang-Low-Chiang [7] in which heterogenous congestion control protocols operate on shared links. When the sources which share the links use a homogenous congestion control protocol which react to the same pricing signals, then the resulting equilibrium is shown to be the solution to an optimization problem that maximizes the sum of user utilities (cf. Kelly et al. [3], Low [6]). When sources use heterogenous congestion control protocols as in Internet, the equilibrium properties of the network are not as well understood. Tang-Wang-Low-Chiang [7] is a first attempt to study the existence, uniqueness, and stability properties of equilibria in network settings with heterogenous protocols. In this section, we use Theorem 2 to prove the uniqueness of equilibrium under weaker requirements than in [7].

Consider a network consisting of a set of links $L = \{1, 2, .., n\}$ with finite capacities c_l. Each link has a price p_l as its congestion measure. Let the set $J = \{1, 2, .., |J|\}$ denote the set of protocols each of which react differently to the price of a link. The effective price for protocol j on link l is denoted by a function $m_l^j(p_l)$ of the price on the link. For each protocol j, there are N^j sources using protocol j. Each protocol j has an $L \times N^j$ routing matrix associated with it denoted by R^j. We have $(R^j)^{ls} = 1$ if source s of type j uses link l, and 0 otherwise. The effective price observed by a source s of type j is

$$q_s^j(p) = \sum_{l \in L} (R^j)^{ls} m_l^j(p_l). \tag{22}$$

Each source s of type j has a strictly increasing and differentiable utility function u_s^j associated with it and sends an amount of traffic given by

$$x_s^j(p) = \mathrm{argmax}_{z \geq 0} u_s^j(z) - z q_s^j(p) = \left[(u_s^j)'^{-1}(q_s^j(p)) \right]^+ .$$

Then, the flow on a link l is given by

$$y^l(p) = \sum_{j \in J} \sum_{s \in N^j} (R^j)^{ls} x_s^j(p). \tag{23}$$

Price p is a network equilibrium if the flow on each link is no more than the link capacity, and is equal to the link capacity if the price on the link is strictly positive. Thus, the equilibrium set is characterized by

$$E = \{p \in \mathbb{R}^n \mid p \geq 0,\ p_l(y^l(p) - c_l) = 0 \text{ and } y^l(p) \leq c_l,\ \forall\, l \in L = \{1, 2, .., n\}\}. \tag{24}$$

Tang-Wang-Low-Chiang show uniqueness properties for the equilibrium given by (22)-(24) under the following assumptions.

(A1) Utility functions u_s^j are strictly concave, increasing, and twice continuously differentiable. Price mapping functions m_l^j are continuously differentiable and strictly increasing with $m_l^j(0) = 0$.
(A2) For any $\epsilon > 0$, there exists a scalar p^ϵ such that if $p_l > p^\epsilon$ for link l, then $x_s^j(p) < \epsilon$ for all (j, s) with $(R^j)^{ls} = 1$.

(A3) For every link l, there exists a source which sends flow only on link l and whose utility function satisfies $(u_s^j)'(c_l) > 0$.

Under (A3), it can be seen that for all equilibrium vectors $p \in E$, p_l is strictly positive for all $l \in L$, implying that all links operate at capacity. In this section, we relax the restrictive assumption (A3) and extend the uniqueness results of [7] to allow for equilibria in which some links in the network possibly operate below capacity.

Under (A1) and (A2), it can be seen that there exists a scalar $p^{\max}$ such that

$$y_l(p) < c_l \tag{25}$$

for every $l \in L$ and p such that $p_l \geq p^{\max}$ (cf. Lemma 1 in [7]). This shows, in particular, that $E \subset M$ where M is the box constrained region given by

$$M = \{p \in \mathbb{R}^n \mid 0 \leq p_l \leq p^{\max}, \ \forall\, l \in L\}.$$

Let $F : U \mapsto \mathbb{R}^n$ be the function given by

$$F_l(p) = c_l - y_l(p).$$

We claim that, under (A1) and (A2),

$$E = \mathrm{Cr}(F, M). \tag{26}$$

First let $p \in E$. We have $p \in M$ since $E \subset M$. Then, $p \in \mathrm{Cr}(F, M)$ in view of the definitions given by (2)-(4) and (24). Conversely, let $p \in \mathrm{Cr}(F, M)$. By Eq. (25), $F_l > 0$ for all l such that $p_l = p^{\max}$. Then, for each $l \in L$, either Eq. (2) or Eq. (3) holds, showing that $p \in E$ in view of the definition in (24). Thus we have $E = \mathrm{Cr}(F, M)$ as claimed. Given $p \in M$, let

$$I^{NB}(p) = \{l \in L \mid 0 < p_l < p^{\max}\} \text{ and } I^F(p) = \{l \in L \mid F_l(p) = 0\}.$$

It can be seen that the function F is continuously differentiable (cf. [7]). Given $p \in E$, we define the index of the equilibrium $p \in E$ to be

$$I(p) = \mathrm{sign}\left(\det\left(\nabla F(p)|_{I^{NB}(p)}\right)\right).$$

We introduce the following complementarity assumption.

(A4) For every $p \in E$ and $l \in L$, $F_l(p) = 0$ only if $p_l > 0$. In other words, for every $p \in E$, $I^{NB}(p) = I^F(p)$.

Any network which satisfies (A3) also satisfies (A4) but not vice versa. The following theorem, therefore, is a generalization of Theorem 4 in [7] in which (A3) is replaced by (A4).

Theorem 3 Consider a network given by (22)-(24). Assume that the network is *regular*, i.e. $I(p) \neq 0$ for all $p \in E$. Further, assume that the network satisfies the assumptions (A1),(A2), and (A4). Then, E has a finite number of elements and

$$\sum_{p \in E} I(p) = 1.$$

Proof. Every $p \in \mathrm{Cr}(F, M)$ is non-degenerate since the network is regular and is complementary by (A4). Then, the result follows from Theorem 1 since (A1) and (A2) imply $E = \mathrm{Cr}(F, M)$ [cf. Eq. (26)].

The following uniqueness result follows from the preceding Theorem and generalizes Theorem 6 in [7].

Theorem 4 Consider a network given by (22)-(24). Assume that the network is regular and satisfies the assumptions (A1),(A2), and (A4). Further, assume that $I(p) = 1$ for all $p \in E$. Then E has a unique element.

The following theorem follows from Theorem 2 and generalizes Theorem 4 by removing (A4) which is difficult to establish algebraically.

Theorem 5 Consider a network given by (22)-(24). Assume that the network satisfies assumptions (A1) and (A2). Further, assume that $\nabla F(p)|_{I^F(p)}$ is a P-matrix for every $p \in E$. Then E has a unique element.

Using the above theorem, we can establish sufficient conditions on the fundamentals of the network for the uniqueness of equilibrium. The following result generalizes Theorem 7 in [7] by relaxing assumption (A3).

Theorem 6 Consider a network given by (22)-(24). Assume that the network satisfies assumptions (A1) and (A2). Denote the derivatives of the price mapping functions with

$$(m_l^j)' = \frac{\partial m_l^j(p_l)}{\partial p_l}.$$

E has a unique element if the functions $(m_l^j)'$ satisfy either one of the following conditions:
1. $\forall l \in L$, there exists some $\mu_l > 0$ such that,

$$(m_l^j)' \in [\mu_l, 2^{1/n}\mu_l], \ \forall\ j \in J.$$

2. $\forall j \in J$, there exists some $\mu^j > 0$ such that,

$$(m_l^j)' \in [\mu^l, 2^{1/n}\mu^j], \ \forall\ l \in L.$$

Proof. Assume that the $(m_l^j)'$ functions satisfy Condition 1. We claim that $\nabla F(p)$ is a P-matrix for any $p \in M$. Tang-Wang-Low-Chiang [7] show that $\det(\nabla F(p)) > 0$ for all p (cf. Corollary 8 in [7]). Let $K \subset L$ be any subset of the links, $p \in M$ be a fixed price vector and consider the principal sub-matrix $S(p) = \nabla F(p)|_K$. We claim that the same proof in [7] shows $\det(S(p)) > 0$. It can be seen that

$$\nabla F(p) = \sum_{j \in J} B^j M^j.$$

where, M^j denotes the $n \times n$ diagonal matrix with diagonal entries $(m_l^j)'(p_l)$ and B^j denotes the $n \times n$ matrix with entries

$$(B^j)^{kl} = \sum_{s \in N_j} (R^j)^{ks} (R^j)^{ls} \left(- \frac{\partial^2 u_s^j\left(x_s^j(p)\right)}{\partial \left(x_s^j\right)^2} \right)^{-1}.$$

Since M^j is a diagonal matrix, we have

$$S(p) = \nabla F(p)|_K = \sum_{j \in J} B^j|_K M^j|_K.$$

In other words, $S(p)$ has the same structure as the complete matrix $\nabla F(p)$, with only the index set L replaced by the index set $K \subset L$. Then, the proof given in [7] shows that $\det(S(p)) > 0$ if the functions $(m_l^j)'$ satisfy

$\tilde{1}$. For any $j \in J$, there exists some $\mu_l > 0$ such that

$$(m_l^j)' \in [\mu_l, 2^{1/|K|}\mu_l].$$

$|K| \leq n$ implies $2^{1/n} \leq 2^{1/|K|}$ and therefore Condition 1 implies Condition $\tilde{1}$, showing $\det(S(p)) > 0$. Since determinants of arbitrary principal sub-matrices of $\nabla F(p)$ are positive, $\nabla F(p)$ is a P-matrix for all $p \in M$. Then, $\nabla F(p)|_{I^F(p)}$ is also a P-matrix for all $p \in E$, which, by Theorem 5, implies that E has a unique element. The proof for the case when m_l^j satisfy Condition 2 can likewise be given. **Q.E.D.**

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