Existence of Global Minima for Constrained Optimization[1]

A. E. Ozdaglar[2] and P. Tseng[3]

Communicated by A. Miele

[1] We thank Professor Dimitri Bertsekas for his comments and support in the writing of this paper.

[2] Assistant Professor, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts.

[3] Professor, Department of Mathematics, University of Washington, Seattle, Washington.

Abstract. We present a unified approach to establishing the existence of global minima of a (non)convex constrained optimization problem. Our results unify and generalize previous existence results for convex and nonconvex programs, including the Frank-Wolfe theorem, and for (quasi)convex quadratically constrained quadratic programs and convex polynomial programs. For example, instead of requiring the objective/constraint functions to be constant along certain recession directions, we only require them to linearly recede along these directions. Instead of requiring the objective/constraint functions to be convex polynomials, we only require the objective function to be a (quasi)convex polynomial over a polyhedral set and the constraint functions to be convex polynomials or the composition of coercive functions with linear mappings.

Key Words. Solution existence, global minima, constrained optimization, recession directions, convex polynomial functions.

1 Introduction

Consider the constrained optimization problem

$$\begin{array}{rl} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0, \;\; i = 1, ..., r, \end{array} \tag{P}$$

where $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper lower semicontinuous (lsc) functions. We are interested in conditions on $f_0, f_1, ..., f_r$ under which a global minimum of (P) exists. In what follows, we denote by D the feasible set of (P), i.e.,

$$D = \text{dom} f_0 \cap C, \qquad C = \cap_{i=1}^r C_i, \qquad C_i = \{x \mid f_i(x) \le 0\}, \;\; i = 0, 1, ..., r, \tag{1}$$

with $\text{dom} f = \{x \mid f(x) < \infty\}$ for any $f : \Re^n \to (-\infty, \infty]$.

The existence question has been studied extensively in the case where (P) is a convex program, i.e., $f_0, f_1, ..., f_r$ are convex (Refs. 1–5, in particular, Section 2.3.1 of Ref. 4). In particular, Luo and Zhang (Ref. 5, Theorem 3) showed that (P) has a global minimum whenever $f_1, ..., f_r$ are convex quadratic and f_0 is quasiconvex quadratic over a polyhedral set. By adapting the proof idea of Luo and Zhang, Auslender (Ref. 2, Theorem 2), proved existence of global minimum of (P) when $f_0, f_1, ..., f_r$ are convex, have the same effective domain, and each belongs to the class $\mathcal{F}$ defined in Ref. 2. This result is further expanded in Refs. 3–4. The result of Luo and Zhang was also extended by Belousov and Klatte (Ref. 6) to the case where $f_0, f_1, ..., f_r$ are convex polynomials. However, for nonconvex programs, the results have been less complete (Refs. 2–3, 7–9, in particular, Section 3.4 of Ref. 3). Auslender (Ref. 2, Theorem 3; also see Ref. 3, Corollary 3.4.3) showed that if f_0 belongs to $\mathcal{F}$ and the feasible set C is *asymptotically linear* and the horizon function of f_0 is nonnegative on the horizon cone of C, then (P) has a global minimum. However, the assumption of C being asymptotically linear excludes cases such as when $f_1, ..., f_r$ are convex quadratic. Thus, these existence results for nonconvex programs do not generalize the aforementioned results for convex programs.

In this paper, we present an approach to establishing the existence of global minimum of (P) that unifies and generalizes some of the approaches in Refs. 2 and 4–6, in particular, Section 2.3 of Ref. 4, for convex programs, and the approaches in Refs. 2–3, 7–8, for nonconvex programs. In particular, in Section 3, we prove our main result, which asserts the existence of a global minimum under key properties about the asymptotic behavior of the functions f_i along "directions of unboundedness" of the corresponding level sets. In the following sections, we use this result to prove the existence of global minima for many different classes of nonconvex and convex programs. In particular, we consider nonconvex programs in which the functions f_i have the form

$$f_i(x) = h_i(A_i x) + b_i^T x + c_i, \tag{2}$$

where h_i is an lsc function having certain coercivity properties. The above functions may be viewed as nonconvex generalizations of convex quadratic functions. We next consider several problem classes studied in Refs. 5–6, 8 and prove existence under more general assumptions. In particular, instead of requiring f_0 to be constant along certain recession directions as in Ref. 8, Theorem 21, we only require f_0 to linearly recede along these directions. Instead of requiring f_0 to be quadratic over a polyhedral set and $f_1, ..., f_r$ to be quadratic as in Ref. 5, we only require f_0 to be quasiconvex polynomial over a polyhedral set and $f_1, ..., f_r$ to have the form (2); see Section 5. Instead of requiring f_0 to be a convex polynomial as in Ref. 6, we only require f_0 to be quasiconvex polynomial over a polyhedral set; see Sections 4 and 5. We also use our main result to deduce, as a direct consequence, the Frank-Wolfe theorem (Ref. 10); see Section 6 of this paper. The notion of a function retracting along "asymptotically nonpositive" directions will play a key role in our analysis.

Regarding notation, all vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors x and y. We denote the 2-norm $\|x\| = (x^T x)^{1/2}$ and ∞-norm $\|x\|_\infty = \max_i |x_i|$, where x_i denotes the ith component of x. For any proper lsc function $f : \Re^n \to (-\infty, \infty]$, we denote, for each $\gamma \in \Re$, the γ-level set as

$$\text{lev}_f(\gamma) = \{x \mid f(x) \le \gamma\}.$$

We write $\gamma^k \downarrow 0$ when the sequence γ^k approaches 0 monotonically from above.

2 Functions Retracting along Asymptotically Nonpositive Directions

The existence of a global minimum of (P) is closely related to properties of the level sets of the functions f_i and "directions of unboundedness" for these sets. In particular, when (P) has a finite minimum value which we assume without loss of generality to be zero, there exists a sequence $x^k \in C$, $k = 1, 2, ...$, with $f_0(x^k) \downarrow 0$. Existence of global minima amounts to f_0 having certain asymptotic linearity properties along each direction d that is a cluster point of $\{x^k/\|x^k\|\}$ when $\|x^k\| \to \infty$. We define these notions below. The first two definitions play a key role in the analysis of Auslender, among others (Refs. 2–3, 7–8).

Definition 2.1 For any set $S \subseteq \Re^n$, its *horizon cone* S_∞ (Ref. 11, Section 3B), also called *asymptotic cone* in Ref. 3, is defined to be

$$S_\infty = \left\{ d \mid \exists\, x^k \in S,\ t^k \to \infty,\ \text{with}\ \frac{x^k}{t^k} \to d \right\}.$$

When S is a closed convex set, we have

$$S_\infty = \{ d \mid \exists\, x \in S,\ \text{with}\ x + \alpha d \in S\ \forall\ \alpha \geq 0 \},$$

and moreover S_∞ is convex.

Definition 2.2 For any function $f : \Re^n \to (-\infty, \infty]$, let f_∞ be the *horizon function* (Ref. 11, p. 88), also called the *asymptotic function* in Ref. 3, defined by

$$f_\infty(d) = \liminf_{\substack{h \to d \\ t \to \infty}} \frac{f(th)}{t}.$$

We say that $d \in \Re^n$ is a *recession direction* of f (Ref. 8, p. 7) if

$$f_\infty(d) \leq 0.$$

The *recession cone* of f is defined to be $R_f = \{d \mid f_\infty(d) \leq 0\}$.

We next introduce the notion of a direction along which a function asymptotically approaches below zero. Such directions will play a key role in our analysis.

Definition 2.3 For any function $f : \Re^n \to (-\infty, \infty]$, we say that $d \in \Re^n$ is an *asymptotically nonpositive direction* (AND) of f if there exists a sequence $x^k \in \mathrm{dom} f$, $k = 1, 2, ...$, satisfying

$$\limsup_{k \to \infty} f(x^k) \leq 0, \quad \|x^k\| \to \infty, \quad \text{and} \quad \frac{x^k}{\|x^k\|} \to d.$$

We refer to $\{x^k\}$ as a *unbounded sequence* (US) associated with d.

Notice that every AND is a recession direction of f with unity norm. If f is convex and $\inf_x f(x) \leq 0$, then the converse is also true. This is because there exists $\{y^k\} \subseteq \Re^n$ with $\lim_{k\to\infty} \sup f(y^k) \leq 0$, so that, for any recession direction d of f with unity norm, the sequence $x^k = y^k + t^k d$, with $t^k = \|y^k\|^2 + k$, is a US associated with d.

We will develop conditions for each f_i, based on its behavior along any AND d, which collectively will ensure existence of global minima for (P). Thus, the question of existence is decomposed into questions about whether each f_i has certain properties along an AND d. In particular, the following two properties along an AND will play a key role in our analysis.

Definition 2.4 For any function $f : \Re^n \to (-\infty, \infty]$ and any $d \in \Re^n$, we say that f *recedes below 0* along d on a set $S \subseteq \text{dom} f$ if, for each $x \in S$, there exists $\bar{\alpha} \geq 0$ such that

$$f(x + \alpha d) \leq 0 \quad \forall\, \alpha \geq \bar{\alpha}.$$

Definition 2.5 For any function $f : \Re^n \to (-\infty, \infty]$ and any AND d of f, we say that f *retracts* along d on a set $S \subseteq \text{dom} f$ if, for any US $\{x^k\} \subseteq S$ associated with d, there exists $\bar{k}$ such that

$$f(x^k - d) \leq \max\{0, f(x^k)\} \quad \forall\, k \geq \bar{k}. \tag{3}$$

In the case of $S = \text{dom} f$, we say that f *retracts* along d. We define f to *retract strongly* along d in the same way except that "$\max\{0, f(x^k)\}$" is replaced by "$f(x^k)$" in (3).

Roughly speaking, the above definition says that one can retract from x^k along d to stay in the level set whenever x^k is sufficiently far from the origin. Notice that any AND d of f is an AND of $\max\{0, f\}$ and vice versa. Moreover,

$$\begin{array}{lll} f \text{ retracts strongly along } d & \Rightarrow & f \text{ retracts along } d \\ & \Leftrightarrow & \max\{0, f\} \text{ retracts strongly along } d. \end{array}$$

The next two examples show that neither of the preceding two notions implies the other.

Example 2.1 Let

$$f(x,y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } y = 0, \\ 0 & \text{if } x = 0 \text{ and } y = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then f is proper, lsc, and $d = (1,0)$ is an AND. Moreover, f retracts along d, but f does not recede below 0 along d on $\text{lev}_f(0)$.

Example 2.2 Let

$$f(x,y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } |y| \leq \sqrt{x}, \\ 1 & \text{otherwise.} \end{cases}$$

Then f is proper, lsc, and $d = (1,0)$ is the only AND of f. f recedes below 0 along d on $\text{dom}f$; however, f does not retract along d. In particular, the sequence $\{(k, \sqrt{k})\}$ is a US associated with $(1,0)$ for which the relation (3) does not hold for any $\bar{k}$.

The notion of function retraction is closely related to the class $\mathcal{F}$ of functions defined in Ref. 2, Definition 7, also called *asymptotically level stable* functions in Ref. 3, p. 94. Specifically, a proper lsc function $f : \Re^n \to (-\infty, \infty]$ belongs to $\mathcal{F}$ if, for any $\alpha > 0$, any convergent sequence $\{\epsilon^k\} \subset \Re$, any sequence $\{x^k\} \subseteq \Re^n$, and any $d \in \Re^n$ satisfying

$$x^k \in \text{lev}_f(\epsilon^k), \qquad \|x^k\| \to \infty, \qquad \frac{x^k}{\|x^k\|} \to d, \qquad f_\infty(d) = 0,$$

there exists $\bar{k}$ such that

$$x^k - \alpha d \in \text{lev}_f(\epsilon^k) \quad \forall\ k \geq \bar{k}.$$

Various examples of functions in $\mathcal{F}$, such as convex piecewise linear-quadratic functions and asymptotically linear functions, are given in Ref. 2. The following lemma shows that $f \in \mathcal{F}$ implies f retracts along any AND whose horizon function value is 0.

Lemma 2.1 If $f \in \mathcal{F}$, then, for any AND d of f, either f retracts along d or $f_\infty(d) < 0$.

Proof. Fix any AND d of f. Then $d \neq 0$ and $f_\infty(d) \leq 0$. If $f_\infty(d) < 0$, the proof is complete. Suppose that $f_\infty(d) = 0$. For any US $\{x^k\}$ associated with d, let $\epsilon^k = \max\{0, f(x^k)\}$. Then $x^k \in \mathrm{lev}_f(\epsilon^k)$ for all k and $\epsilon^k \to 0$, so $f \in \mathcal{F}$ implies there exists $\bar{k}$ such that, for all $k \geq \bar{k}$, $x^k - d \in \mathrm{lev}_f(\epsilon^k)$ or, equivalently, $f(x^k - d) \leq \max\{0, f(x^k)\}$. $\square$

In general, $f_\infty(d) < 0$ does not imply f retracts along d. In his earlier work (Ref. 7, p. 777 and Ref. 8, Theorem 13), Auslender considered a larger class of functions which, roughly speaking, allow different α for different k. However, this class is too large to apply to the problem (P) since one needs a common α to work for the objective function and the constraint functions. Hence the class $\mathcal{F}$ was introduced in Ref. 2 and, analogously, we introduce the notion of a function retracting along an AND (which uses a common $\alpha = 1$). We will discuss the class $\mathcal{F}$ further in Section 7.

3 Main Existence Result

Define

$$L(\gamma) = \mathrm{lev}_{f_0}(\gamma) \cap C,$$

where C is given by (1). Below, we prove our main result on the existence of a global minimum of (P). This amounts to showing that $L(\gamma) \neq \emptyset$ for all $\gamma > 0$ implies $L(0) \neq \emptyset$. The proof, by induction on r, uses similar ideas as in the proof of Theorem 3 in Ref. 5 and Theorem 1 in Ref. 2 (also see Proposition 1.5.7 in Ref. 4), but we deal with more general functions than convex quadratic or convex functions belonging to $\mathcal{F}$. The following key assumption on the functions will be used:

(A1) **(a)** For each $i \in \{0, 1, ..., r\}$ and each AND d of f_i,
either (i) f_i recedes below 0 along d on $\mathrm{dom} f_i$,
or (ii) f_i retracts along d and, in case $r \neq 0$, f_i recedes below 0 along d on C_i.

(b) In case $r \neq 0$, $C_0 \subseteq \mathrm{dom} f_i$ for $i = 1, ..., r$.

Proposition 3.1 (Existence of Global Minima of (P)). Suppose that $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper, lsc, and satisfy Assumption A1. Also, suppose that $L(\gamma) \neq \emptyset$ for all $\gamma > 0$. Then $L(0) \neq \emptyset$.

Proof. Take any sequence of scalars $\{\gamma^k\}_{k=1,2,...} \downarrow 0$. By assumption, $L(\gamma^k) \neq \emptyset$ for all k. Since f_i is lsc for all i, $L(\gamma^k)$ is closed. Let

$$x^k \in \arg\min\{\|x\| \mid x \in L(\gamma^k)\}.$$

If x^k has a convergent subsequence, then the limit point of this subsequence would, by lsc of f_i for all i, be in $L(0)$. Thus, it suffices to consider the case of $\|x^k\| \to \infty$.

Since $\{x^k/\|x^k\|\}$ is bounded, it has a subsequence converging to some limit $d \neq 0$. By passing to a subsequence if necessary, we can assume that $x^k/\|x^k\| \to d$. Since $f_0(x^k) \leq \gamma^k$ while $f_i(x^k) \leq 0$ for $i = 1, ..., r$, this implies that d is an AND of f_i for $i = 0, 1, ..., r$. Then, by Assumption A1(a), for each $i \in \{0, 1, ..., r\}$, either f_i recedes below 0 along d on domf_i or f_i retracts along d. Moreover, if $r \neq 0$, then f_i recedes below 0 along d on C_i for $i \in \{0, 1, ..., r\}$.

If f_0 recedes below 0 along d on domf_0, then for any $\bar{x} \in D \neq \emptyset$, we have $\bar{x} \in$ domf_0 so that $f_0(\bar{x} + \alpha d) \leq 0$ for all α sufficiently large. In case $r \neq 0$, for each $i \in \{1, ..., r\}$ we have $\bar{x} \in C_i$ and hence $f_i(\bar{x} + \alpha d) \leq 0$ for all α sufficiently large. Thus, $\bar{x} + \alpha d \in L(0)$ for all α sufficiently large, implying $L(0) \neq \emptyset$. Thus, it remains to consider the case of f_0 retracting along d. If f_i retracts along d for all $i = 1, ..., r$, then for each $i \in \{0, 1, ..., r\}$, there exist $\bar{k}_i$ such that

$$x^k - d \in \text{lev}_{f_i}(\gamma^{k,i}) \quad \forall\ k \geq \bar{k}_i,$$

where $\gamma^{k,0} = \gamma^k$ and $\gamma^{k,i} = 0$ for $i = 1, ..., r$. Then, for all $k \geq \max_{i=0,1,...,r} \bar{k}_i$,

$$\tilde{x}^k = x^k - d$$

would satisfy $\tilde{x}^k \in L(\gamma^k)$. Then, by $x^k/\|x^k\| \to d$, we would have $d^T x^k/\|x^k\| \to \|d\|^2$, which implies $d^T x^k \to \infty$ and hence

$$\|\tilde{x}^k\|^2 = \|x^k\|^2 - 2d^T x^k + 1 < \|x^k\|^2$$

for all k sufficiently large, contradicting x^k being an element of $L(\gamma^k)$ of least 2-norm. Thus, either $L(0) \neq \emptyset$ or else $r \geq 1$ and $f_{\bar{i}}$ recedes below 0 along d on dom$f_{\bar{i}}$ for some $\bar{i} \in \{1, ..., r\}$.

We now complete the proof by induction on r. The above argument shows that Proposition 3.1 is true when $r = 0$. Suppose Proposition 3.1 is true for $r = 0, 1, ..., \bar{r}$ for some $\bar{r} \geq 0$. Consider $r = \bar{r} + 1$. We showed above that either $L(0) \neq \emptyset$ or there exists some AND d of $f_0, f_1, ..., f_r$ such that $f_{\bar{i}}$ recedes below 0 along d on $\mathrm{dom} f_{\bar{i}}$ for some $\bar{i} \in \{1, ..., r\}$. Let us drop $f_{\bar{i}}$ from $L(\gamma)$ to obtain

$$\tilde{L}(\gamma) = \mathrm{lev}_{f_0}(\gamma) \bigcap \left(\bigcap_{\substack{i=1 \\ i \neq \bar{i}}}^{r} C_i \right).$$

Then $\tilde{L}(\gamma^k) \supset L(\gamma^k) \neq \emptyset$ for all k. Also, Assumption A1 is still satisfied upon dropping $f_{\bar{i}}$. Thus, by the induction hypothesis, $\tilde{L}(0) \neq \emptyset$. For any $\tilde{x} \in \tilde{L}(0)$, since f_i recedes below 0 along d on C_i for all $i \in \{0, 1, ..., r\} \setminus \{\bar{i}\}$, we have $\tilde{x} + \alpha d \in \tilde{L}(0)$ for all $\alpha \geq 0$ sufficiently large. Moreover, Assumption A1(b) implies $\tilde{x} \in \mathrm{lev}_{f_0}(0) = C_0 \subseteq \mathrm{dom} f_{\bar{i}}$ so $f_{\bar{i}}$ receding below 0 along d on $\mathrm{dom} f_{\bar{i}}$ implies $f_{\bar{i}}(\tilde{x} + \alpha d) \leq 0$ for all α sufficiently large, yielding $\tilde{x} + \alpha d \in L(0)$. $\square$

Proposition 3.1 shows that when $f_0, f_1, ..., f_r$ satisfy Assumption A1 and the minimum value of (P) is finite which we assume without loss of generality to be zero, a global minimum of (P) exists. We remark on Proposition 3.1 and its assumptions in more detail below.

(i) It can be seen from its proof that Proposition 3.1 still holds if we relax Assumption A1(b) so that $C_0 \subseteq \mathrm{dom} f_i$ holds only for those $i \in \{1, \ldots, r\}$ for which there exists some AND d such that f_i does not retract along d. For instance, if every constraint function f_i is polyhedral, then it can be shown that f_i retracts along every AND of f_i, so that Assumption A1(b) can be dropped altogether. In general, Assumption A1(b) cannot be dropped as we will show in Example 4.1.

(ii) Proposition 3.1 still holds if we relax the assumption on f_0 within Assumption A1(a) to

For each AND d of f_0, there exists $F \subseteq \mathrm{dom} f_0$ such that

(i) f_0 recedes below 0 along d on $\mathrm{dom} f_0 \setminus F$,

(ii) f_0 retracts along d on F,

(iii) In case $r \neq 0$, f_0 recedes below 0 along d on C_0.

[The assumption on f_0 within Assumption A1(a) corresponds to $F = \emptyset$ or $F = \mathrm{dom} f_0$.] This is because in the proof we can analogously divide into two cases: If $D \setminus F \neq \emptyset$, then for any $\bar{x} \in D \setminus F$ we have $f_0(\bar{x} + \alpha d) \leq 0$ for all α sufficiently large. Otherwise $D \subseteq F$, and the proof proceeds as before. This seemingly technical generalization of Proposition 3.1 is needed for Sections 5 and 6.

The following example shows that the assumption on the constraint functions $f_1, ..., f_r$ cannot be relaxed in the same way.

Example 3.1 Let

$$f_0(x,y) = \begin{cases} \frac{1}{|y|+1} - \frac{1}{2} & \text{if } |y| < 1 \text{ and } x \geq 0, \\ 0 & \text{if } |y| \geq 1 \text{ and } x \geq 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$f_1(x,y) = \begin{cases} 0 & \text{if } |y| \leq 1 - e^{-x} \text{ and } x \geq 0, \\ 1 & \text{if } |y| > 1 - e^{-x} \text{ and } x \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Then, both f_0 and f_1 are proper, lsc. Any d with $\|d\| = 1$ and $d_1 \geq 0$ is an AND of f_0. It can be seen that for any AND d of f_0, f_0 retracts along d, and also f_0 recedes along d on C_0. The only AND of f_1 is $d = (0,1)$, and f_1 does not recede below 0 along d on $\text{dom} f_1$. Moreover, the sequence $\{(k, 1 - e^{-k})\}$ is a US associated with $d = (0,1)$, which does not satisfy (3), hence f_1 does not retract along d. But f_1 does satisfy the following relaxed assumption (which is analogous to the relaxed assumption on f_0): Let $F_1 = \{(x,y) \mid x \geq 0, |y| \geq 1\}$. Then f_1 recedes below 0 along d on $\text{dom} f_1 \setminus F_1$, retracts along d on F_1, and f_1 recedes below 0 along d on C_0. Although the point $(x^k, y^k) = (k, 1 - e^{-k})$ satisfies $f_1(x^k, y^k) = 0$ and $f_0(x^k, y^k) \to 0$, there is no solution to $f_1(x,y) \leq 0$ and $f_0(x,y) \leq 0$.

(iii) If, for each $i = 1, ..., r$ and each AND d of the constraint function f_i, f_i retracts along d and recedes below 0 along d on C_i, then Proposition 3.1 still holds if the second part of Assumption A1(a)(ii) on the objective function f_0, namely "in case $r \neq 0$, f_0 recedes below 0 along d on C_0", is dropped. In fact, in this case, (P) can be reduced to the case of $r = 0$ by working with the extended objective function

$$\tilde{f}_0(x) = \begin{cases} f_0(x) & \text{if } x \in D, \\ \infty & \text{otherwise} \end{cases}$$

It can be shown that, for this equivalent problem, Assumption A1(a) is satisfied. This device will be used in the proof of Lemma 6.1.

The following example shows that the second part of Assumption A1(a)(ii) on f_0 cannot be dropped in general.

Example 3.2 Let

$$f_0(x,y) = \begin{cases} \frac{1}{|y|+1} & \text{if } x > 0 \text{ and } |y| < 2\sqrt{x}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_1(x,y) = \begin{cases} 0 & \text{if } x \geq 1 \text{ and } |y| \leq \sqrt{x}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, both f_0 and f_1 are proper, lsc. Moreover, any d with $\|d\| = 1$ is an AND of f_0 and f_0 retracts along d. The only AND of f_1 is $d = (1,0)$ and f_1 recedes below 0 along d on domf_1. Also, the point $x^k = k, y^k = \sqrt{k}$ satisfies $f_1(x^k, y^k) = 0$ and $f_0(x^k, y^k) = 1/(\sqrt{k}+1) \to 0$. However, there is no solution to $f_1(x,y) \leq 0$ and $f_0(x,y) \leq 0$. The problem is that f_0 does not recede below 0 along $d = (1,0)$ on C_0.

If the objective function f_0 retracts strongly along each AND d of $f_0, f_1, ...f_r$ and each constraint function f_i retracts along d, then the existence of a global minimum of (P) can be deduced without assuming its minimum value is finite. This result generalizes an existence result of Auslender (Ref. 7, Theorem 2.4); see Proposition 7.3.

Proposition 3.2 Consider the problem (P). Suppose that $D \neq \emptyset$ and, for each AND d of $f_0, f_1, ...f_r$, we have that f_0 retracts strongly along d and f_i retracts along d for $i = 1, ..., r$. Then (P) has a global minimum.

Proof. Let γ^* denote the minimum value of (P), which we assume without loss of generality to be either 0 or $-\infty$. Then, for any sequence $\gamma^k \downarrow \gamma^*$, $L(\gamma^k) \neq \emptyset$ for all k. Let x^k be an element of $L(\gamma^k)$ of minimum 2-norm. We claim that $\{x^k\}$ has a cluster point, which would be a global minimum of (P). We argue this by contradiction. Suppose that $\|x^k\| \to \infty$. By passing to a subsequence if necessary, we assume that $x^k/\|x^k\|$ converges to some d. Then d is an AND of $f_0, f_1, ..., f_r$. Since f_0 retracts strongly along d, $f_0(x^k - d) \leq f_0(x^k) \leq \gamma^k$ for all k. For each $i \in \{1, ..., r\}$, since f_i retracts along d and $x^k \in C_i$, $f_i(x^k - d) \leq \max\{0, f_i(x^k)\} \leq 0$ for all k sufficiently large. Thus $x^k - d \in L(\gamma^k)$ for all k sufficiently large. But also $\|x^k - d\|^2 < \|x^k\|^2$ for all k sufficiently large (see the proof of Proposition 3.1), contradicting x^k being of minimum 2-norm. $\square$

4 Linearly Receding Functions

In this section we introduce an important class of functions that satisfy Assumption A1(a). We say that f *linearly recedes* along d if there exists $\theta \in \Re$ (depending on d) such that $\theta \leq 0$ and

$$f(x + \alpha d) = f(x) + \alpha\theta \quad \forall\ \alpha \in \Re,\ \forall\ x \in \mathrm{dom} f. \tag{4}$$

We say that f is *constant* along d if we can take $\theta = 0$ in the above expression. Consider the following assumption on a proper lsc $f : \Re^n \to (-\infty, \infty]$:

(A2) For any AND d of f, f linearly recedes along d.

Assumption A2 says that f is affine on each line intersecting $\mathrm{dom} f$ and parallel to an AND of f. This assumption is weaker than f belonging to the class $\mathcal{F}_1$ defined in Ref. 2, Definition 3, namely, f is convex and f linearly recedes along any recession direction of f. [Recall that any AND of f is a recession direction of f.] This assumption is also weaker than f being *asymptotically directionally constant* (Ref. 3, p. 86), namely, f is constant along any recession direction of f. We will show in Lemma 4.3 that there is a large class of nonconvex functions f that satisfy Assumption A2 but are not asymptotically directionally constant nor in $\mathcal{F}_1$.

The following lemma shows that it suffices to verify Assumption A2 for each f_i in lieu of Assumption A1(a).

Lemma 4.1 If $f_0, f_1, ..., f_r$ each satisfies Assumption A2, then they satisfy Assumption A1(a).

Proof. Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ satisfying Assumption A2. Fix any $d \in \Re^n$ that is an AND of f. By Assumption A2, f linearly recedes along d, i.e., there exists $\theta \in \Re$ such that $\theta \leq 0$ and (4). If $\theta < 0$, then for any $x \in \mathrm{dom} f$, we have from (4) that $f(x + \alpha d) \to -\infty$ as $\alpha \to \infty$. Thus f recedes below 0 along d on $\mathrm{dom} f$. If $\theta = 0$, then for any $x \in \mathrm{dom} f$, we have from (4) that $f(x - \alpha d) = f(x)$ for all $\alpha \in \Re$. It follows that f retracts along d and f recedes below 0 along d on $\mathrm{lev}_f(0)$.

Applying the above result to $f_0, f_1, ..., f_r$, we see that they satisfy Assumption A1(a). $\square$

Using Lemma 4.1, we give below an example showing that Proposition 3.1 is false if Assumption A1(b) is dropped.

Example 4.1 Define

$$\phi(t) = \begin{cases} -\log t - \log(1-t) & \text{if } 0 < t < 1, \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$f_0(x_1, x_2) = x_1, \qquad f_1(x_1, x_2) = \phi(x_1) - x_2.$$

Then f_0 is linear, so f_0 satisfies Assumption A2. Since ϕ is lsc, then f_1 is lsc and, in fact, convex. Since ϕ has bounded support, it is not difficult to see that, for any $d = (d_1, d_2) \in \Re^2$ that is an AND of f_1, we have $d_1 = 0$ and $d_2 \geq 0$. Then f_1 linearly recedes along d. Thus f_0, f_1 each satisfies Assumption A2. By Lemma 4.1, they satisfy Assumption A1(a).

As $x_1 \to 0^+$ and setting $x_2 = \phi(x_1)$, we have that $f_0(x_1, x_2) \to 0$ and $f_1(x_1, x_2) \leq 0$. However, there is no $x \in \Re^2$ satisfying $f_0(x) = 0$ and $f_1(x) \leq 0$ (since $f_1(x) \leq 0$ implies $x_1 > 0$). Here, the problem is that domf_1 is not contained in the 0-level set of f_0 (in fact, the two sets are disjoint), thus violating Assumption A1(b).[4]

The following lemma shows that the assumption $\theta \leq 0$, implicit in Assumption A2, is redundant whenever f does not tend to $-\infty$ at a superlinear rate. In particular, convex functions have the latter property.

Lemma 4.2 Suppose that $f : \Re^n \to (-\infty, \infty]$ is proper, and

$$f_\infty(d) > -\infty \quad \forall\, d \in \Re^n. \tag{5}$$

If an AND d of f and $\theta \in \Re$ satisfies (4), then $\theta \leq 0$.

Proof. Fix any AND d of f and $\theta \in \Re$ that satisfies (4). If $\theta = 0$, then clearly $\theta \leq 0$. If $\theta \neq 0$, then fix any $\bar{x} \in$ dom$f$, which exists since f is proper. Since d is an AND of f, there exists an associated US $\{x^k\}$. For each k, let $t^k = \|x^k - \bar{x}\|$ and $d^k = (x^k - \bar{x})/\|x^k - \bar{x}\|$. Then $d^k \to d$ and

$$x^k = y^k + t^k d, \quad \text{with} \quad y^k = \bar{x} + t^k(d^k - d).$$

[4] We can make these two sets have nonempty intersection by redefining $\phi(t)$ to have the value 1 (instead of ∞) for all $t \leq -1$, say. However, the resulting f_1 is no longer convex. If $f_0, f_1, ..., f_r$ are convex, can Assumption A1(b) be relaxed to $C_0 \cap (\cap_{i=1}^r \text{dom} f_i) \neq \emptyset$? This is an open question.

For each k, since $x^k \in \mathrm{dom} f$, (4) implies $y^k \in \mathrm{dom} f$ and

$$f(x^k) = f(y^k + t^k d) = f(y^k) + t^k \theta.$$

Thus

$$\theta = \frac{f(x^k) - f(y^k)}{t^k}. \tag{6}$$

Notice that $\{f(x^k)\}$ is bounded above and, by $\|d^k - d\| \to 0$ and f satisfying (5), we have

$$\liminf_{k \to \infty} \frac{f(y^k)}{t^k} = \liminf_{\substack{k \to \infty \\ d^k \neq d}} \frac{f(\bar{x} + t^k(d^k - d))}{t^k \|d^k - d\|} \|d^k - d\| \geq 0.$$

If $d^k = d$ for k along a subsequence, then $y^k = \bar{x}$ so $f(y^k)/t^k \to 0$ along this subsequence. Then the limit superior of the right-hand side of (6) as $k \to \infty$ is nonpositive, implying $\theta \leq 0$. □

If $f : \Re^n \to (-\infty, \infty]$ is convex and $\inf_x f(x) \leq 0$, then we saw in Section 2 that

$$d \text{ is an AND of } f \quad \Leftrightarrow \quad d \in R_f.$$

In this case, the assumption that f is constant along any AND d of f (i.e., θ can be taken equal to 0 in (4)) is equivalent to $R_f \subseteq L_f$, where $L_f = R_f \cap (-R_f)$ denotes the constancy space of f. Since $L_f \subseteq R_f$, this is equivalent to

$$R_f = L_f.$$

Similarly, it can be seen that f linearly recedes along d at rate θ if and only if $(d, \theta) \in L_{\mathrm{epi} f}$, where $\mathrm{epi} f$ denotes the epigraph of f and $L_S = S_\infty \cap (-S_\infty)$ denotes the lineality space of a set $S \subseteq \Re^{n+1}$. Thus, Assumption A2 is equivalent to

$$R_f \subseteq \mathrm{Proj}_{\Re^n} L_{\mathrm{epi} f},$$

where $\mathrm{Proj}_{\Re^n}(x, \zeta) = x$ for all $(x, \zeta) \in \Re^n \times \Re$. Some related conditions are given in Ref. 4, Proposition 2.3.4.

We now give examples of functions that satisfy Assumption A2. Consider the following assumption on f.

(A3) $$f(x) = h(Ax) + b^T x + c,$$

where $A \in \Re^{m \times n}, b \in \Re^n, c \in \Re$ and $h : \Re^m \to (-\infty, \infty]$ is a function satisfying (5) and

either (i) $b = 0$, $\liminf_{\|y\| \to \infty} h(y) = \infty$ or (ii) $b \neq 0$, $\liminf_{\|y\| \to \infty} \frac{h(y)}{\|y\|} = \infty$.

Assumption A3 is satisfied by, for example, convex quadratic functions and the function $g(Ax-b)$ in Ref. 8, p. 13 with g coercive (i.e., $g_\infty(d) > 0$ for all $d \in \Re^n$). We show below that Assumption A3 implies Assumption A2.

Lemma 4.3 Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper lsc function that satisfies Assumption A3. Then f satisfies Assumption A2.

Proof. First, we claim that

$$d \text{ is an AND of } f \quad \Rightarrow \quad Ad = 0, \quad b^T d \leq 0. \tag{7}$$

To see this, fix any AND d of f and let $\{x^k\}$ be an associated US.

If $b = 0$, then $f(x^k) = h(Ax^k) + c$ is bounded above, so that $\|Ax^k\|$ is bounded. Hence $\|Ax^k\|/\|x^k\| \to 0$, implying $Ad = 0$. Since $b = 0$, we also have $b^T d \leq 0$.

If $b \neq 0$, then we have

$$\lim_{k\to\infty} \sup \frac{h(Ax^k)}{\|x^k\|} + \frac{b^T x^k + c}{\|x^k\|} = \lim_{k\to\infty} \sup \frac{f(x^k)}{\|x^k\|} \leq 0.$$

Thus $h(Ax^k)/\|x^k\|$ is bounded from above. If $\|Ax^k\| \to \infty$ along a subsequence, then boundedness from above of

$$\frac{h(Ax^k)}{\|x^k\|} = \frac{h(Ax^k)}{\|Ax^k\|} \frac{\|Ax^k\|}{\|x^k\|}$$

and $h(Ax^k)/\|Ax^k\| \to \infty$ imply $\|Ax^k\|/\|x^k\| \to 0$ along this subsequence. If $\|Ax^k\|$ is bounded along a subsequence, then $\|x^k\| \to \infty$ implies $\|Ax^k\|/\|x^k\| \to 0$ along this subsequence. In either case, we see that $\|Ax^k\|/\|x^k\| \to 0$ and hence $Ad = 0$. Also, we have from $\{f(x^k)\}$ being bounded above (by, say, κ) that

$$\frac{b^T x^k}{\|x^k\|} \leq -\frac{h(Ax^k)}{\|x^k\|} + \frac{\kappa - c}{\|x^k\|},$$

which, together with

$$\lim_{k\to\infty} \inf \frac{h(Ax^k)}{\|x^k\|} \geq 0, [5]$$

[5] Why? If $\|Ax^k\|$ is bounded along a subsequence, then $h(Ax^k)$ is bounded below along this subsequence (since h is lsc and nowhere equal to $-\infty$) and hence $\dfrac{h(Ax^k)}{\|x^k\|} \to 0$. If

yields

$$\lim_{k\to\infty} \sup \frac{b^T x^k}{\|x^k\|} \le 0,$$

so that $b^T d \le 0$. Thus d satisfies

$$Ad = 0, \quad b^T d \le 0.$$

It follows from $f(x) = h(Ax) + b^T x + c$ and (7) that f satisfies Assumption A2. □

Thus, if each of $f_0, f_1, ..., f_r$ satisfies Assumption A3, then it satisfies Assumption A2. By Lemma 4.1, they collectively satisfy Assumption A1(a). So, if they in addition satisfy Assumption A1(b), then Proposition 3.1 yields that a global minimum of (P) exists. To our knowledge, this existence result is new.

5 Quasiconvex Polynomial Functions over Polyhedral Sets

Consider the following polynomial assumption on f, generalizing the quadratic assumption of Luo and Zhang (Ref. 5, Section 4).

(A4) $$f(x) = \begin{cases} g(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$
with $g : \Re^n \to \Re$ a polynomial function and

$$X = \{x \mid Bx \le b\},$$

for some $B \in \Re^{m\times n}, b \in \Re^m$. Moreover, g is assumed to be quasiconvex on X (Ref. 12, Chap. 9).

Any convex polynomial f satisfies this assumption. A nonconvex example is $g(x_1, x_2) = x_1 x_2$, $X = [0, \infty) \times (-\infty, 0]$. The following lemma shows that f satisfies the relaxed assumption in Remark (ii) following Proposition 3.1.

$\|Ax^k\| \to \infty$ along a subsequence, then the assumption of $h_\infty(d) > -\infty$ for all $d \in \Re^m$ implies $\frac{h(Ax^k)}{\|Ax^k\|}$ is bounded below along this subsequence, which together with $\frac{\|Ax^k\|}{\|x^k\|} \to 0$ implies $\lim_{k\to\infty} \inf \frac{h(Ax^k)}{\|x^k\|} = \lim_{k\to\infty} \inf \frac{h(Ax^k)}{\|Ax^k\|}\frac{\|Ax^k\|}{\|x^k\|} \ge 0$.

Lemma 5.1 Let $f : \Re^n \to (-\infty, \infty]$ satisfy Assumption A4. Then, for any AND d of f, there exists $F \subseteq X$ such that (i) f recedes below 0 along d on $X \setminus F$, (ii) f retracts along d on F, (iii) f recedes below 0 along d on $\mathrm{lev}_f(0)$.

Proof. Fix any AND d of f. Let $\{x^k\}$ be a US associated with d. Let

$$\gamma^k = \sup_{\ell \geq k} \max\{0, f(x^\ell)\} \quad \forall\ k.$$

Then $\{\gamma^k\} \downarrow 0$ and, for each k, $x^\ell \in \mathrm{lev}_f(\gamma^k)$ for all $\ell \geq k$. Since g is lsc quasiconvex on X, the level sets of f are closed convex. Since $\mathrm{lev}_f(\gamma^k)$ is closed convex and $x^\ell/\|x^\ell\| \to d$, d is in the horizon cone of $\mathrm{lev}_f(\gamma^k)$ (Ref. 13). Thus, d is in the intersection of the horizon cones of $\mathrm{lev}_f(\gamma^k)$ for all k. Since the horizon cones of $\mathrm{lev}_f(\gamma^k)$, $k = 1, 2, ...,$ are nested, this implies that d is in the horizon cone of $\mathrm{lev}_f(\gamma)$ for all $\gamma > 0$.

Fix any $x \in X$. Then $x \in \mathrm{lev}_f(\gamma)$ for some $\gamma > 0$, so that $x + \alpha d \in \mathrm{lev}_f(\gamma)$ for all $\alpha \geq 0$. Thus, $B(x + \alpha d) \leq c$ and $g(x + \alpha d) \leq \gamma$ for all $\alpha \geq 0$. The former implies $Bd \leq 0$, i.e., $d \in X_\infty$. Since $g(x+\alpha d)$ is a polynomial function of α, the latter implies that

$$\text{either} \quad g(x + \alpha d) \to -\infty \ \text{ as } \alpha \to \infty \quad \text{or} \quad g(x + \alpha d) = g(x) \ \ \forall\ \alpha \in \Re. \tag{8}$$

Define

$$F = \{x \in X \mid g(x + \alpha d) = g(x)\ \forall\ \alpha \in \Re\}.$$

Then, for any $x \in X \setminus F$, we have from $Bd \leq 0$ that $x + \alpha d \in X$ for all $\alpha \geq 0$, so that (8) yields

$$f(x + \alpha d) = g(x + \alpha d) \to -\infty \quad \text{as} \ \ \alpha \to \infty.$$

This shows that f recedes below 0 along d on $X \setminus F$. Fix any US $\{x^k\} \subseteq F$ associated with d. For each $i \in \{1, ..., m\}$, let B_i and b_i denote the ith row of B and c. If $B_i d = 0$, then $B_i(x^k - d) = B_i x^k \leq b_i$ for all k. If $B_i d < 0$, then it follows from $B_i x^k/\|x^k\| \to B_i d$ and $\|x^k\| \to \infty$ that there exists $\bar{k}_i$ such that

$$B_i(x^k - d) \leq b_i \quad \forall\ k \geq \bar{k}_i.$$

Let $\bar{k} = \max_{\{i | B_i d < 0\}} \bar{k}_i$. Since $x^k \in F$, this yields

$$f(x^k - d) = g(x^k - d) = g(x^k) = f(x^k) \quad \forall\ k \geq \bar{k},$$

implying that f retracts along d on F. By $Bd \leq 0$ and (8), for any $x \in \text{lev}_f(0)$, there exists $\bar{\alpha} \geq 0$ such that

$$x + \alpha d \in X, \qquad g(x + \alpha d) \leq g(x) \leq 0 \quad \forall\, \alpha \geq \bar{\alpha}.$$

Thus $f(x + \alpha d) \leq 0$ for all $\alpha \geq \bar{\alpha}$, so f recedes below 0 on $\text{lev}_f(0)$. □

We note that the above proof generalizes to any continuous function g that is either constant or tends to ∞ or $-\infty$ on each line. In the case where g is quadratic, i.e., $g(x) = \frac{1}{2}x^T Qx + q^T x$ for some symmetric $Q \in \Re^{n\times n}$ and $q \in \Re^n$, it can be seen that, for any AND d of f, either $d^T Qd < 0$, in which case $F = \emptyset$, or else $d^T Qd = 0, (Qx + q)^T d \leq 0$, in which case F is the set of maxima of the linear program

$$\max \;\; (Qx + q)^T d \quad \text{s.t.} \quad x \in X.$$

Thus, F is a face of X in this case. In the special case where f is a convex polynomial function on $\Re^n$, Lemma 5.1 can be further sharpened as is shown below.

Lemma 5.2 Let $f : \Re^n \to \Re$ be a convex polynomial function. Then, for any AND d of f, either (i) f recedes below 0 along d on $\Re^n$ or (ii) f retracts along d on $\Re^n$ and f recedes below 0 along d on $\text{lev}_f(0)$.

Proof. Fix any AND d of f. Then d is a recession direction of f. Since f is convex, this implies that, for any $x \in \Re^n$, $f(x + \alpha d) \leq f(x)$ for all $\alpha \geq 0$ (Refs. 4 and 13). Since $f(x + \alpha d)$ is a polynomial function of α, this implies that

$$\text{either} \quad \lim_{\alpha\to\infty} f(x + \alpha d) = -\infty \quad \text{or} \quad f(x + \alpha d) = f(x) \;\; \forall\, \alpha \in \Re.$$

If $\lim_{\alpha\to\infty} f(x + \alpha d) = -\infty$ for every $x \in \Re^n$, then clearly f recedes below 0 along d on $\Re^n$. Thus, it remains to consider the case where, for some $\bar{x} \in \Re^n$, $f(\bar{x} + \alpha d) = f(\bar{x})$ for all $\alpha \in \Re$. Since f is convex, this implies $d \in L_f$ and hence, for *every* $x \in \Re^n$,

$$f(x + \alpha d) = f(x) \;\; \forall\, \alpha \in \Re.$$

Thus, f retracts along d on $\Re^n$ and f recedes below 0 along d on $\text{lev}_f(0)$ (which may be empty). □

Suppose that f_0 satisfies Assumption A4 and $f_1, ..., f_r$ each satisfies Assumption A3. By Lemma 5.1, f_0 satisfies the relaxed assumption in Remark (ii) following Proposition 3.1. By Lemmas 4.3 and 4.1, $f_0, f_1, ..., f_r$ satisfy Assumption A1(a), subject to the above modification. Thus, if they also satisfy Assumption A1(b), then by Proposition 3.1 and Remark (ii) following it, (P) has a global minimum whenever its minimum value is finite. This result generalizes Corollary 2 and Theorem 3 in Ref. 5 which further assume g and $f_1, ..., f_r$ are quadratic. Even in the special case of f_0 being convex polynomial and $f_1, ..., f_r$ being convex quadratic, our result appears to be new. Unlike the proof in Ref. 5, our proof does not rely on the canonical form of a quasiconvex quadratic function over a convex set.

Suppose that f_0 satisfies Assumption A4 and $f_1, ..., f_r$ are convex polynomial functions on $\Re^n$. By Lemma 5.1, f_0 satisfies the relaxed assumption in Remark (ii) following Proposition 3.1. By Lemma 5.2, $f_0, f_1, ..., f_r$ satisfy Assumption A1, subject to the above modification. Thus, by Proposition 3.1 and Remark (ii) following it, (P) has a global minimum whenever its minimum value is finite. This result generalizes Theorem 3 in Ref. 6, which further assumes f_0 to be convex on $\Re^n$.

Notice that, because g is quasiconvex only on X, we cannot treat $Bx \leq c$ as constraints but, rather, must incorporate it into the objective function. Then we exploit the fact that a recession direction for a polyhedral set can be retracted from points in the set that are sufficiently far out.

6 The Frank-Wolfe Theorem

In this section we assume that f_0 is a quadratic function over a polyhedral set, as was studied by Frank and Wolfe (Ref. 10) and many others; see Refs. 5–6 for more detailed discussions.

(A5)
$$f(x) = \begin{cases} g(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

with $g(x) = \frac{1}{2}x^T Q x + q^T x$ for some symmetric $Q \in \Re^{n\times n}$ and $q \in \Re^n$, and

$$X = \{x \mid Bx \leq b\},$$

for some $B \in \Re^{m\times n}, b \in \Re^m$.

The following lemma shows that f_0 satisfies the relaxed assumption in Remark (ii) following Proposition 3.1 for $r = 0$.

Lemma 6.1 Let $f : \Re^n \to (-\infty, \infty]$ satisfy Assumption A5. Then, for any AND d of f, there exists $F \subseteq X$ such that (i) f recedes below 0 along d on $X \setminus F$, (ii) f retracts along d on F.

Proof. Fix any AND d of f. Let $\{x^k\}$ be a US associated with d. Then $x^k \in X$ for all k and $\lim_{k\to\infty} \sup g(x^k) \le 0$. Then

$$0 \ge \lim_{k\to\infty} \sup \frac{g(x^k)}{\|x^k\|^2} = \lim_{k\to\infty} \sup \frac{\frac{1}{2}(x^k)^T Q x^k + q^T x^k}{\|x^k\|^2} = \frac{1}{2} d^T Q d.$$

Thus $d^T Q d \le 0$. Also, $d \in X_\infty$ (see the proof of Lemma 5.1).

Define

$$F = \{x \in X \mid \nabla g(x)^T d \ge 0\}.$$

For any $x \in X \setminus F$, since $d^T Q d \le 0$, we have

$$g(x + \alpha d) = g(x) + \alpha \nabla g(x)^T d + \frac{1}{2}\alpha^2 d^T Q d \to -\infty \quad \text{as} \quad \alpha \to \infty.$$

Since $d \in X_\infty$, we also have $x + \alpha d \in X$ for all $\alpha \ge 0$. Thus f recedes below 0 along d on $X \setminus F$.

Also, we have using $d^T Q d \le 0$ that

$$g(x - d) = g(x) - \nabla g(x)^T d + \frac{1}{2} d^T Q d \le g(x) \quad \forall\, x \in F.$$

Since X is polyhedral, we have that $x^k - d \in X$ for all k sufficiently large (see the proof of Lemma 5.1). This implies that if $\{x^k\} \subseteq F$, then $f(x^k - d) \le f(x^k)$ for all k sufficiently large. Thus, f retracts along d on F. □

Suppose that f_0 satisfies Assumption A5 and $r = 0$. By Lemma 6.1, f_0 satisfies the relaxed assumption in Remark (ii) following Proposition 3.1. Thus, by Proposition 3.1 and Remark (ii) following it, (P) has a global minimum whenever its minimum value is finite. This is the classical Frank-Wolfe theorem (Ref. 10).

In Section 1 of Ref. 6, it is mentioned that Andronov et al. (Ref. 14) had extended the Frank-Wolfe theorem to the case of a cubic function over a polyhedral set. It is also known that this result does not extend to polynomial of degree 4 or higher (Ref. 10). Can this result be deduced from Proposition 3.1 similarly as the Frank-Wolfe theorem?

7 Further Applications

In this section, we present further applications of Proposition 3.1 and Lemma 4.1 and indicate their connection to existing results.

Following Refs. 2–3, we say that a closed set $S \subseteq \Re^n$ is *asymptotically linear* if

$$\delta_S \in \mathcal{F},$$

where δ_S is the indicator function for S, i.e.,

$$\delta_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{otherwise.} \end{cases}$$

An example of such a set S is the Minkowski sum of a compact set with a finite collection of polyhedral sets. The level set of a convex quadratic function is generally not asymptotically linear. We have as a corollary of Proposition 3.1 and Lemma 2.1 the following refinement of Ref. 2, Theorem 3 and Ref. 3, Corollary 3.4.3.

Proposition 7.1 Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ and any closed asymptotically linear set $S \subseteq \Re^n$ such that $S \cap \mathrm{dom} f \neq \emptyset$. Suppose that $\inf_{x \in S} f(x) = 0$ and

$$f_\infty(d) \geq 0 \quad \forall\, d \in S_\infty.$$

Suppose also that either $f \in \mathcal{F}$ or f is constant along any AND of f. Then there exists an $x^* \in S$ with $f(x^*) = 0$.

Proof. Define

$$f_0(x) = f(x) + \delta_S(x) \quad \forall\, x \in \Re^n.$$

Then f_0 is proper, lsc, and $\inf_x f_0(x) = 0$. Moreover, $(f_0)_\infty(d) \geq f_\infty(d)$ for $d \in S_\infty$ and $(f_0)_\infty(d) = \infty$ otherwise. Thus, our assumption on $f_\infty(d)$ implies

$$(f_0)_\infty(d) \geq 0 \quad \forall\, d \in \Re^n. \tag{9}$$

Also, for any $\epsilon \in \Re$, $x \in \mathrm{lev}_{f_0}(\epsilon)$ if and only if $x \in \mathrm{lev}_f(\epsilon)$ and $x \in S$. For any nonzero $d \in \Re^n$, $(f_0)_\infty(d) = 0$ implies $f_\infty(d) = (\delta_S)_\infty(d) = 0$.

Suppose that $f \in \mathcal{F}$. Since f and δ_S are in $\mathcal{F}$, the above observations show that f_0 is in $\mathcal{F}$. Then, by Lemma 2.1, (9), and Proposition 3.1 with $r = 0$, there exists an $x^* \in \Re^n$ with $f_0(x^*) = 0$.

Suppose instead that f is constant along any AND of f. Fix any AND d of f_0 and any associated US $\{x^k\}$. Then d is an AND of f, so that f is constant along d. Also, $x^k \in S$ for all k, implying $(\delta_S)_\infty(d) = 0$. Since $\delta_S \in \mathcal{F}$, this implies that there exists a $\bar{k}$ such that $x^k - d \in S$ for all $k \geq \bar{k}$. Also, $f(x^k - d) = f(x^k)$ for all k. Thus $f_0(x^k - d) = f_0(x^k)$ for all $k \geq \bar{k}$. This shows that f_0 retracts along any AND of f_0. By Proposition 3.1 with $r = 0$, there exists an $x^* \in \Re^n$ with $f_0(x^*) = 0$. $\square$

If S is further assumed to be asymptotically multipolyhedral, then instead of f being constant along any AND of f, it suffices that f linearly recedes along any AND of f (see Section 4). The next lemma shows that if $f_0, f_1, ..., f_r$ are convex and in $\mathcal{F}$, then they satisfy Assumption A1(a).

Lemma 7.1 Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ such that $f \in \mathcal{F}$ and f is convex. Then, for each AND d of f,

either (i) f recedes below 0 along d on $\mathrm{dom} f$

or (ii) f retracts along d and f recedes below 0 along d on $\mathrm{lev}_f(0)$.

Proof. Fix any AND d of f. Then $f_\infty(d) \leq 0$. Since f is convex, this implies that f recedes below 0 along d on $\mathrm{lev}_f(0)$. If $f_\infty(d) < 0$, then the convexity of f implies that f recedes below 0 along d on $\mathrm{dom} f$. If $f_\infty(d) = 0$, then Lemma 2.1 implies that f retracts along d. $\square$

Lemma 7.1 and Proposition 3.1 together yield the following existence result of Auslender (Ref. 2, Theorem 2).

Proposition 7.2 Suppose that $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper, lsc, convex, and belong to $\mathcal{F}$. Also, suppose that $\mathrm{dom} f_0 = \mathrm{dom} f_i$, $i = 1, ..., r$, and $L(\gamma) \neq \emptyset$ for all $\gamma > 0$. Then $L(0) \neq \emptyset$.

If a proper lsc function $f : \Re^n \to (-\infty, \infty]$ satisfies Assumption A2 and (5), then the indicator of its 0-level set,

$$f^\circ(x) = \begin{cases} 0 & \text{if } f(x) \leq 0, \\ \infty & \text{otherwise,} \end{cases}$$

satisfies (5) (since $(f^\circ)_\infty(d) \geq 0$ for all $d \in \Re^n$) but may not satisfy Assumption A2. In particular, take $f(x) = \|Ax\|^2 + b^T x$ for some A, b with $b \neq 0$. Then, we know from Section 4 that f satisfies Assumption A2 and (5). However, for any $d \in \Re^n$ such that $Ad = 0$ and $b^T d < 0$, it is easily seen that d is an AND of f°, but f° does not linearly recede along d (since $f^\circ(x+\alpha d)=0$ for all $\alpha \geq 0$ while $f^\circ(x+\alpha d)=\infty$ for all α sufficiently negative). Thus, we cannot incorporate constraints into the objective by means of an indicator function and still satisfy Assumption A2 for the objective function f_0.

In Ref. 7, Theorem 2.4 (also see Ref. 8, Theorem 21), Auslender considered the following problem

$$\min \quad g(x) \quad \text{s.t.} \quad x \in X, \tag{10}$$

where X is an asymptotically multipolyhedral set in $\Re^n$ and $g : \Re^n \to (-\infty, \infty]$ is a proper lsc function. Auslender showed that if

$$\text{dom} g \cap X \neq \emptyset, \qquad g_\infty(d) \geq 0 \quad \forall\, d \in X_\infty, \tag{11}$$

and either (i) g is constant along each $d \in X_\infty \cap R_g$ or (ii) epi$g$ is an asymptotically multipolyhedral set or (iii) g is *weakly coercive*, then (10) has a global minimum. Notice that (iii) implies (i). This is because g being weakly coercive means that $g_\infty(d) \geq 0$ for all $d \in \Re^n$ and, for any $d \in R_g$, g is constant along d.

Auslender's existence result generalizes one of Rockafellar (Ref. 13, Theorem 27.3) for the case where X is a polyhedral set, g is convex, and g is constant along each $d \in X_\infty \cap R_g$. In the case where g and X are convex, it can be seen that (10) having finite minimum value implies (11), but not conversely (e.g., $g(x) = -\log(x)$ if $x > 0$ and $g(x) = \infty$ otherwise satisfies $g_\infty(d) \geq 0$ for all $d \in \Re$). Thus, in this case, the assumption (11) is weaker than assuming (10) has a finite minimum value. In general, neither implies the other. We show below that Auslender's result may be viewed as a special case of Proposition 3.2.

Proposition 7.3 Consider the problem (10). Suppose that (11) holds and either (i) g linearly recedes along each $d \in X_\infty \cap R_g$ or (ii) epi$g$ is an asymptotically multipolyhedral set. Then (10) may be reformulated in the form of (P) while satisfying the assumptions of Proposition 3.2. Hence (10) has a global minimum.

Proof. Suppose that g linearly recedes along each $d \in X_\infty \cap R_g$. Since X is asymptotically multipolyhedral, $X = S + K$, where S is a compact set in $\Re^n$ and $K = \cup_{j=1}^{\ell} K^j$ with each K^j being a polyhedral cone. Then, $x \in X$ if and only if $x = x_1 + x_2$ with $x_1 \in S$ and $x_2 \in K$, so (10) is equivalent to

$$\min \quad f_0(x_1, x_2) \quad \text{s.t.} \quad f_1(x_1, x_2) \leq 0,$$

where

$$\begin{aligned} f_0(x_1,x_2) &= \begin{cases} g(x_1+x_2) & \text{if } x_1 \in S, \\ \infty & \text{otherwise,} \end{cases} \\ f_1(x_1,x_2) &= \begin{cases} 0 & \text{if } x_2 \in K, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that f_0 and f_1 are proper, lsc. We now verify that this problem satisfies the assumptions of Proposition 3.2, and hence it has a global minimum. By (11), $\text{dom} g \cap X \neq \emptyset$, so that $D = \text{dom} f_0 \cap C_1 \neq \emptyset$. Fix any AND (d_1, d_2) of f_0, f_1. Then this is a recession direction of f_i, i.e., $(f_i)_\infty(d_1,d_2) \leq 0$, for $i = 1, 2$. We have

$$\begin{aligned} (f_0)_\infty(d_1,d_2) &= \liminf_{\substack{d_1' \to d_1, d_2' \to d_2 \\ t \to \infty}} \frac{f_0(td_1', td_2')}{t} \\ &= \begin{cases} \liminf_{\substack{d_1' \to 0, d_2' \to d_2 \\ t \to \infty}} \frac{g(t(d_1'+d_2'))}{t} & \text{if } d_1 = 0, \\ \infty & \text{otherwise,} \end{cases} \\ &= \begin{cases} g_\infty(d_2) & \text{if } d_1 = 0, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly,

$$(f_1)_\infty(d_1,d_2) = \begin{cases} 0 & \text{if } d_2 \in K, \\ \infty & \text{otherwise.} \end{cases}$$

This implies $d_1 = 0$, $d_2 \in K$, and $g_\infty(d_2) \leq 0$. Thus, $d_2 \in X_\infty \cap R_g$ and our assumption on g implies $g_\infty(d_2) \geq 0$ and g linearly recedes along d_2. The latter implies there exists $\theta \leq 0$ such that

$$g(x + \alpha d_2) = g(x) + \alpha\theta \quad \forall\, \alpha \in \Re,\ \forall\, x \in \text{dom} g.$$

Since $g_\infty(d_2) \geq 0$, then $\theta = 0$. This and $d_1 = 0$ imply that

$$\begin{aligned} f_0((x_1,x_2) + \alpha(d_1,d_2)) &= g(x_1 + x_2 + \alpha d_2) \\ &= g(x_1 + x_2) \\ &= f_0(x_1,x_2) \qquad \forall\, \alpha \in \Re,\ \forall\, (x_1,x_2) \in \text{dom} f_0. \end{aligned}$$

Thus f_0 is constant (and hence retracts strongly) along (d_1, d_2). Since (d_1, d_2) is an AND of f_1, there exists a US $\{(x_1^k, x_2^k)\}$ such that $x_2^k \in K$ for all k, $\|(x_1^k, x_2^k)\| \to \infty$, and $(x_1^k, x_2^k)/\|(x_1^k, x_2^k)\| \to (d_1, d_2)$. Since $d_1 = 0$, then

$\|x_2^k\| \to \infty$ and $x_2^k/\|x_2^k\| \to d_2$. Since $d_2 \in K$ and K is the union of polyhedral cones, there exists $\bar{k} \geq 0$ such that

$$x_2^k - d_2 \in K \quad \forall\, k \geq \bar{k}$$

(see the proof of Lemma 5.1). Then

$$f_1((x_1^k, x_2^k) - (d_1, d_2)) = f_1(x_1^k, x_2^k - d_2) = 0 \quad \forall\, k \geq \bar{k}.$$

Thus f_1 retracts along (d_1, d_2).

Suppose that epi$g$ is an asymptotically multipolyhedral set. Then (10) is equivalent to

$$\min \quad h(x, \mu) \quad \text{s.t.} \quad (x, \mu) \in Y,$$

where $h(x, \mu) = \mu$ and $Y = \text{epi}g \cap (X \times \Re)$. The assumption (11) implies dom$h \cap Y \neq \emptyset$ and $h_\infty(d, \delta) = \delta \geq 0$ for all $(d, \delta) \in Y_\infty$. Also, $\delta \leq 0$ for any $(d, \delta) \in R_h$, which implies $\delta = 0$ so that h is constant along (d, δ). It can be shown that the intersection of two asymptotically multipolyhedral sets is also asymptotically multipolyhedral. Thus case (ii) reduces to case (i). □

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