Competing over Networks

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Recent advances in information technology have allowed firms to gather vast amounts of data regarding consumers' preferences and the structure and intensity of their social interactions. This paper examines a game-theoretic model of competition between firms, which can target their marketing budgets to individuals embedded in a social network. We provide a sharp characterization of the optimal targeted marketing strategies and highlight their dependence on the underlying social network structure. Furthermore, we identify network structures for which the returns to targeting are maximized, and we provide conditions under which it is optimal for the firms to asymmetrically target a subset of the individuals. Finally, we provide a lower bound on the extent of asymmetry in these asymmetric equilibria and therefore shed light on the effect of the network structure to the outcome of marketing competition between firms.

Key words: Social networks, competition, advertising, targeting.

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1. Introduction

It is widely accepted that word-of-mouth plays a central role in the propagation of brand or product information, and thus it is a first order consideration in the design and implementation of a marketing strategy. Moreover, marketers nowadays have access to and can take advantage of vast amounts of data on the pattern and intensity of social interactions between consumers. The advent of the Internet as a prominent communication and advertising platform has enabled firms to implement targeted marketing campaigns and direct their efforts to certain subsets of the population. The natural question that arises in this setting is how a firm can use the wealth of available information along with the targeting technologies to increase the awareness about its products. The recent acquisitions of Buddy Media, Vitrue, and Wildfire Interactive by Salesforce, Oracle, and Google respectively point to the indisputable fact that social media marketing emerges as a viable alternative to traditional advertising and tech giants are striving to obtain a competitive advantage in the new landscape.

The focus in this paper is on prescribing the best way a firm can exploit word-of-mouth and its knowledge over the social network structure of consumers when devising a targeted marketing campaign. At the core of the model lies an *information externality* that arises endogenously due to communication: information obtained by an agent in the network can be passed along to her peer group and thus word of mouth communication may amplify the effect of a firm's marketing efforts. As a consequence, an optimal targeting strategy may involve allocating disproportionate fraction of the advertising budget to certain agents that play a central role in the word of mouth process expecting that they will pass the relevant information to the rest. Indeed, strategies of similar flavor have been applied in practice, however in an ad hoc and heuristic way.[1] Our goal is to provide a systematic characterization of optimal targeting strategies and identify qualitative insights that lead to their success.

We develop a novel model of targeted marketing in the presence of competition and explore its impact on the evolution of *brand awareness* for the competing firms. The level of brand awareness for a firm, defined as the fraction of consumers that name the firm first when asked about a product category,[2] evolves dynamically as agents communicate with their peers over time. Firms can influence the word-of-mouth communication process by targeting advertising funds to specific individuals. Our choice of the level of brand awareness in the population as the firm's metric of performance for a marketing campaign is motivated by the fact that growing brand awareness is the primary goal of devising a marketing campaign especially in a new market. Moreover, awareness is much easier to measure and relate to marketing efforts and it is positively correlated with sales and profits.

The word-of-mouth process along with the marketing strategies of the firms define a dynamical system that tracks the evolution of the brand awareness levels for the competing firms. Our goal is to characterize the limiting behavior of this system as a function of the underlying network structure and the marketing strategies. First, we show that our objective is well defined since the agents' awareness levels converge to a fixed vector. Next, we provide a sharp characterization of the limit as a function of the underlying network structure and the advertising efforts of the firms.

[1] For example, Klout (www.klout.com), a San Francisco-based company, provides social media analytics to measure a user's influence in her social network. The service scrapes social network data and assigns individuals a "Klout" score, which presumably reflects their influence. Then, it connects businesses with individuals of high score with the intention of influencing the latter to spread good publicity for the former in exchange for free merchandise and other perks.

[2] There are three classical measures of brand awareness in any given product category: spontaneous awareness that measures the fraction of consumers that indicate they know the brand without any prompting, top-of-mind awareness that measures the fraction of consumers that name the brand *first*, and finally aided awareness where consumers indicate whether they know brands presented to them in a list. Although there are obvious differences between the three measures, they are very closely related (see Laurent et al. (1995)). Our analysis mostly focuses on the top-of-mind awareness metric, however we believe that our insights remain valid for the other brand awareness metrics.

It turns out that the average brand awareness converges (almost surely) to a weighted sum of the advertising funds that firms allocate to individuals, where the weights are given by a notion of *centrality* in the underlying network structure. This notion of centrality can be best understood in terms of a random walk on a graph that consists of the underlying social network structure and a set of nodes corresponding to the competing brands. Agents are central if they are visited frequently by the random walk before it hits any of the nodes representing the brands. Armed with a complete characterization of the limiting behavior of the dynamical system, we proceed to the problem of devising optimal marketing strategies for the firms over the social network structure. To this end, we show that the fraction of a firm's advertising budget that is targeted to an individual in an optimal allocation is an increasing function of her centrality in the network.

In the second part of the paper we model the competition between two firms as a game over the network of agents. First, firms simultaneously choose their marketing strategies, i.e., how to allocate their advertising budgets to the agents. Secondly, agents obtain information both from their peers and the firms over time until their awareness levels converge to a limiting vector. We are interested in characterizing the equilibrium strategies for the firms and derive qualitative insights about the relation of the network structure with the level of competition between them. We show that in this ex-ante symmetric environment, a symmetric equilibrium in which firms target the same individuals always exists. However, we also show the possibility of asymmetric equilibria in which the two firms target different subsets of individuals and we provide explicit conditions on the dynamics of the information exchange process and the network structure under which asymmetric equilibria are guaranteed to exist.

As a way of further illustrating the effectiveness of targeting technologies, we study a setting where only one of the firms can use targeting whereas the other allocates its budget uniformly to the agents. In this setting, we compare the returns from advertising for the first firm in the case when it has perfect knowledge over the social network structure and uses its targeting technologies with the case when the firm chooses not to use targeting and simply allocates its advertising budget uniformly to the agents. We provide a characterization of the difference in the levels of brand awareness for the firm in these two cases as a function of the underlying network structure and highlight the relation of the success of a targeted marketing strategy with the level of heterogeneity in the network interactions among agents: the more homogeneous these interactions are, the less effective a targeted marketing strategy is compared to one that is agnostic on the underlying network.

Finally, we turn our attention to the setting where both of the firms have access to targeting technologies and examine the extent to which equilibrium behavior leads to differences in the effectiveness of their marketing campaigns even when they have equal budgets. To this end, we provide a bound on the maximum value that the ratio between the awareness levels of the competing firms at equilibrium can take over all networks. Unlike the previous result, i.e., the returns to targeting - when only one of the firms can use targeting - are maximized for heterogeneous structures, we obtain a different insight: markets that feature large asymmetries in the outcome of the marketing competition, exhibit heterogeneity among agents with *sufficiently high* centrality. In particular, the extreme case is a barbell network with two highly central nodes (albeit with different centralities).

A novel feature of our model is that we introduce competition between firms over a social network structure. To the best of our knowledge, there is very little work that attempts to capture the interaction between competition and network structure. Most closely related to our paper is the recent contribution by Goyal and Kearns (Goyal and Kearns (2012)) who examine a contagion model where two competing firms simultaneously choose their seed sets and then agents adopt one of two technologies according to a stochastic diffusion process related to the general threshold model. Their main goal is to study the level of inefficiency in the use of available resources, i.e., number of seeds, at equilibrium. They also introduce the notion of a *budget multiplier* that measures the extent to which ex-ante imbalances in players' budgets get amplified at equilibrium and identify a certain property on the adoption dynamics which guarantees that the budget multiplier remains bounded. In contrast to their paper, our focus is on explicitly characterizing the impact of the social network structure on the equilibrium outcomes of the competition between the firms and on the resulting dynamics of their levels of brand awareness as well as on prescribing optimal (network-dependent) targeted marketing strategies. Fazeli and Jadbabaie (2012) consider a similar setting as Goyal and Kearns (2012) with the difference that agents determine which technology to adopt based on the outcome of a local coordination game with their peers and they provide a lower and an upper bound on the proportion of product adoptions.

There is a sizable literature on awareness formation models that describe the growth and decay of a brand's awareness level over time as a function of advertising efforts (see for example Dodson and Muller (1978), Mahajan et al. (1984), Naik et al. (1998), Bass et al. (2007)). In these models, the target market is divided at any point in time into two segments: the *aware* and the *unaware*. Change in the level of brand awareness is driven by the amount of advertising efforts directed to the unaware segment, word-of-mouth communication, and "forgetting" effects. Recently, Naik et al. (2008) extended this model by incorporating competition among brands. In particular, in their model the change in the awareness level for a firm also depends on the advertising efforts of its competitors. Naik et al. (2008) allow for the latter dependence to be both negative (a brand's awareness is decreasing in the competitors' advertising efforts) and positive (to incorporate "confusion effects" (see Pauwels (2004))). Unlike these models which assume a continuum of agents, we consider a society envisaged as a social network of n agents and study the effectiveness of *targeted* marketing campaigns in the presence of competition that explicitly take into account the underlying network structure.

Our work is also related to a recent stream of papers that examine the dynamics of opinion formation and information exchange among agents that are embedded in a social network. DeGroot (1974) introduced a tractable framework to study the interaction among agents in which their *beliefs* about an underlying state (e.g., the quality of a product) are modeled as continuous variables and communication with one's peers takes the form of a simple linear update. On the other extreme, Acemoglu et al. (2012) consider a dynamical model of information exchange among Bayesian agents. Their goal is to identify conditions under which dispersed information is aggregated through the communication process and agents "learn" the true underlying state.

Our model builds on the *binary voter model* which was introduced independently in Holley and Liggett (1975) and Clifford and Sudbury (1973). In the voter model the state (opinion) of an individual is a binary variable and each time communication takes place, the agent simply adopts the opinion of the communicating party. The goal of that literature is to characterize the limiting behavior of the model under specific network structures, e.g., infinite lattices. In recent work, Yildiz et al. (2012) consider an extension where a subset of agents are "stubborn", i.e., their opinion is fixed to one of the two values. Their goal is twofold: first, they want to understand the impact of the stubborn agents in the limiting behavior of the model and, secondly, they consider the problem of optimally placing a fixed number of stubborn agents to maximize the impact on the long-run expected opinions of the population. In our model, the state of an individual is a continuous variable (as in DeGroot (1974)), however the update process is not a simple averaging of the states of the neighbors. Instead, we assume that an agent communicates a message that can take a finite number of different values (corresponding to the options available to the agent) and the content of the message depends on the agent's state. Then, the receiving agent updates her own state which keeps track of the proportion of messages that the agent received favoring each of the available options. Our awareness level update process is more general than both the averaging rule used in DeGroot (1974) and the one suggested by the voter model and it can be thought of as bridging the gap between the "naive" learning rules and the often times intractable Bayesian learning.[3]

Our results illustrate a close connection between optimal marketing strategies in the presence of competition and the centrality of agents in the underlying network structure. In particular, we

[3] As mentioned above, learning models that involve Bayesian agents typically assume that one of the available options is superior (however the superior option is ex-ante unknown). Agents have noisy private information that is correlated with which of the options is superior and exchange this information with their peers. The state of an agent, i.e., her belief that a given option is superior, is updated according to Bayes rule, and at any point in time it is increasing in the number of messages that the agent has received in favor of this option.

show that it is optimal for the firms to allocate their advertising funds to agents in proportion to a notion of network centrality that is very closely related to *Bonacich* centrality which is widely used as a sociological measure of influence.[4] Ballester et al. (2006) and more recently Candogan et al. (2012) also note a link between centralities and economic outcomes. Unlike these models that feature exogenous local payoff complementarities and no competition, in our setting externalities among the agents arise *endogenously* due to the word-of-mouth communication process.

Furthermore, there is a recent series of papers that study algorithmic questions related to marketing strategies over social networks. Kempe et al. (2003) discuss optimal seeding strategies, when agents act according to a pre-specified rule of thumb. They examine both the *linear threshold model*, where an agent adopts a behavior or buys a product if at least a certain fraction of her neighbors do the same, and the *independent cascade model*, where an adopter *infects* (causes the adoption) of a peer with certain, exogenously specified probability. The goal is to identify the seed set that maximizes the eventual set of adopters. In Hartline et al. (2008), a monopolist makes take-it-or-leave-it offers to a sequence of agents for a product that exhibits network effects, i.e., the utility of an agent that purchases the good depends on past sales. They show that although the problem of deriving the optimal pricing strategy is NP-hard, a simple policy that offers the product for free to a strategically chosen subset of agents and then charges the myopically optimal price to the rest, achieves a constant factor approximation of the optimal revenues.

Finally, there is some recent work that identifies the merits of targeted advertising. For example, Iyer et al. (2005) show that that firms find it optimal to advertise more to consumers who have a strong preference for their product rather than to comparison shoppers who can be attracted to the competition. They also show that targeted advertising increases equilibrium profits and it can be more valuable for firms in the presence of competition than price discrimination. Their model features heterogeneity in consumers' preferences and does not explicitly model the word-of-mouth exchange of information over a network, which is the focus of our paper.

The rest of paper is organized as follows. Section 2 introduces the model. Section 3 provides a characterization of the limiting behavior of the dynamical system that captures the information exchange process among the consumers and the firms. Sections 4 and 5 study the marketing budget allocation decisions of the two firms and identify the equilibria of the marketing competition game between them. Sections 6 and 7 illustrate the benefits of targeting and highlight that they are larger in heterogeneous networks. Section 8 discusses how to extend the baseline model by introducing heterogeneity in the consumers' preferences. Finally we conclude in Section 9. To focus on the exposition of our results, we relegate all proofs to the Appendix.

[4] The two notions differ because of the presence of the competing firms in our setting. For a more detailed discussion on the family of centrality measures introduced by Bonacich see Bonacich (1987) and Jackson (2008).

2. Model

The society $\mathcal{G}(\mathcal{N},\mathcal{E})$ consists of a set $\mathcal{N}=\{1,\cdots,n\}$ of agents embedded in a social network represented by the adjacency matrix $\mathcal{E}$. The ij-th entry of $\mathcal{E}$, denoted by e_{ij}, represents the relative frequency of communication of agent i with agent j.[5] We assume that $e_{ij}\in[0,1]$ for all i,j and we normalize $e_{ii}=0$ for all i. We also assume that $\sum_i e_{ij}=1$. Note that we do not impose any symmetry assumption on the e_{ij}'s (although our analysis remains valid in the case when $e_{ij}=e_{ji}$). We investigate the process of awareness formation in a market with two (newly introduced) competing brands, A and B.[6]

The state of the marketing competition at time k can be summarized by the vector of awareness levels for brands A and B at k denoted by $\mathbf{x^A}[k]$ and $\mathbf{x^B}[k]$. In particular, $x_i^A[k]\in[0,1]$ represents the *top-of-mind* awareness of agent i for brand A, i.e., the frequency with which agent i ranks brand A as her first choice in its respective market. Finally, we do not impose that $x_i^A[k]+x_i^B[k]=1$ to allow for the possibility that agent i is unaware of brands A, B (as they may be new in the market) or there is an incumbent brand, the *status quo* (short-handed by SQ), that the agent associates with the market. For this, we let $x_i^{SQ}[k]=1-x_i^A[k]-x_i^B[k]$.

Communication takes place at discrete time periods $k=\{1,2,\dots,\infty\}$. At time period k an agent chosen uniformly at random from the population receives information from her peer group with probability $\alpha\in(0,1)$ and directly from the competing firms with probability $1-\alpha$.[7] In other words, parameter α captures the relative importance of word-of-mouth communication in the process of building brand awareness. The peer group of a given agent i is defined by $\mathcal{N}_i=\{j\in\mathcal{N}|e_{ij}>0\}$. Communication takes the form of a simple message that can take one of three values, A,B, or SQ. When agent i communicates with agent j (which occurs with probability $\frac{1}{n}\alpha e_{ij}$ at time period k), then the message i receives is a function of agent j's state, i.e., the values of $x_j^A[k],x_j^B[k]$, and $x_j^{SQ}[k]$. In particular, a message generating function q maps an agent's state to a message. We assume that:

$$q(x_j^A[k],x_j^B[k],x_j^{SQ}[k])=\begin{cases}A, & \text{with probability } \; x_i^A[k],\\ B, & \text{with probability } \; x_i^B[k],\\ SQ, & \text{with probability } \; 1-x_i^A[k]-x_i^B[k].\end{cases}$$

Finally, if the agent obtains information directly from the firms (which occurs with probability $1-\alpha$), then she receives message A,B or SQ, with probability that is a function of the advertising funds

[5] Alternatively, e_{ij} captures the *strength* of influence of agent j on i.

[6] The analysis readily extends to any finite number of brands. We chose to present our model and results for the case of two to ease the exposition.

[7] The analysis readily extends to the case when agents are heterogeneous with respect to parameter α, i.e., individual i receives information form her peer group with probability α_i. For simplifying the exposition and focusing on the effects we set out to study, we only discuss the case when $\alpha_i=\alpha_j=\alpha$, for all i,j.

that the firms allocate on that individual. In particular, if q_{firms} denotes the message generating function that maps the advertising budgets that the firms allocate to agent i to a message, we have:

$$q_{firms}(b_i(A), b_i(B)) = \begin{cases} A, & \text{with probability } \; h(b_i(A), b_i(B)), \\ B, & \text{with probability } \; h(b_i(B), b_i(A)), \\ SQ, & \text{with probability } \; 1 - h(b_i(A), b_i(B)) - h(b_i(B), b_i(A)), \end{cases}$$

where function h is such that $h : [0,C] \times [0,C] \to [0,1]$ and $h(x,y) + h(y,x) \leq 1$ and C denotes the total marketing budget available to each of the two brands.[8]

At the beginning of the horizon, $k = 0$, the state of the marketing competition is given by arbitrary vectors $\mathbf{x^A}[0]$ and $\mathbf{x^B}[0]$. Let $r[k]$ denote the receiver of the message at time period $k > 0$, $s[k]$ denote the sender of the message, and $m[k]$ denote the content of the message. Then, if agent i receives information at time k and given the history of information exchange $\{r[l], s[l], m[l]\}_{l \leq k}$ up to time k, agent i's awareness of brand A is given by:

$$x_i^A[k] = \frac{\mathbf{1}_{m[k]=A}}{|\{l : r[l] = i, l \leq k\}| + 1} + \frac{|\{l : r[l] = i, l \leq k\}|}{|\{l : r[l] = i, l \leq k\}| + 1} x_i^A[k-1],$$

where $\mathbf{1}_{.}$ is the indicator function of its argument. In other words, $x_i^A[k]$ is updated only when the agent receives information, i.e., $r[k] = i$, otherwise it remains unchanged, and each message that the agent receives is weighted equally. Similarly,

$$x_i^B[k] = \frac{\mathbf{1}_{m[k]=B}}{|\{l : r[l] = i, l \leq k\}| + 1} + \frac{|\{l : r[l] = i, l \leq k\}|}{|\{l : r[l] = i, l \leq k\}| + 1} x_i^B[k-1],$$

and $x_i^{SQ}[k] = 1 - (x_i^A[k] + x_i^B[k])$. Contrast this updating rule with simple averaging that is imposed by consensus models (e.g., DeGroot (1974)). In the latter, later messages gets disproportionally weighted and their contribution to the state at any time k is constant (whereas all messages in our model have equal contribution to the state at time k of the order $1/k$). We are imposing the following assumptions on the model defined above:

Assumption 1. *The communication network defined by adjacency matrix $\mathcal{E}$ is connected, i.e., for every pair of individuals i, j, there exists a communication path from i to j (which may involve multiple steps).*

We further assume the following for function h:

Assumption 2. *Function $h(x,y)$ (recall that $h : [0,C] \times [0,C] \to [0,1]$) is twice continuously differentiable and strictly increasing in x.*

[8] The analysis carries through when the two brands do not have the same marketing budgets. As our goal is to study the effect of the network structure on the marketing competition, we assume that the two brands are ex-ante symmetric.

The two competing firms are interested in maximizing the average awareness of their brands in the population of agents. As we show in the next section, this is a well-defined optimization problem as the state of the marketing competition (i.e., vectors $\mathbf{x^A}, \mathbf{x^B}$) converges (almost surely) in the limit for every allocation of marketing budgets from the firms.

2.1. Discussion of the model

As already mentioned in the introduction, the goal of this paper is to study how information about (new) brands propagates in a social network in the presence of competition. We focus on two channels through which agents obtain information: word of mouth, i.e., communicating with their peers, and direct advertising from the firms. There is plenty of empirical evidence that consumers learn about brands, new products or technologies by exchanging information with their peers as well as by the advertising efforts of firms (see for example Godes and Mayzlin (2004) and Conley and Udry (2010)). The amount of advertising funds that a brand allocates to an individual determines the *intensity* of marketing efforts targeted towards the individual over time. Assumption 2 implies that the returns to advertising, i.e., the change in the level of brand awareness, is increasing with the amount of funds spent on an individual which is natural in our setting. Finally, as will become evident in what follows the two brands are not only competing against each other but they are also competing against the status quo option (SQ). The latter can be interpreted as an incumbent brand that does not engage in targeted marketing and passively allocates its marketing budget uniformly to all the agents. Alternatively, it can be seen as the decay in awareness due to forgetting over time (Mahajan et al. (1984)).

Moreover, we assume that word-of-mouth communication is achieved through the exchange of simple messages that can take one of three values in the benchmark model: A, B, or SQ. The message generating function q imposes the following intuitive assumption in the environment: the higher an agent's awareness level is for option i, the more likely she is to provide information about i. Finally, a consumer's response to the marketing efforts by firms A and B is governed by function h that takes as arguments the advertising funds that A and B allocate on the consumer. This feature of our model is related to the framework of Goyal and Kearns (2012) that study the dynamics of product adoption in the presence of competition. They decompose the adoption process into two parts: *switching* from non-adoption to adoption (that is governed by a function f and takes as an argument the sum of the seeding efforts of the two competitors) and *selecting* which of the two (new) products to adopt (that is governed by a function g). Interpreted in our model, non-adoption is equivalent to the status quo option and our function h is the product of functions f and g (however we do not impose the restriction that h has to be decomposable in this way).

The firms' objective is to maximize the average long-run awareness about their brands. Although we do not directly incorporate purchasing decisions from the agents into the model, we believe that this objective captures the essence of marketing competition between firms, as profits are typically positively correlated to brand awareness and it is often the case that firms invest in marketing not to promote a particular product but rather to build a customer pool that engages in repeat purchases with the firm. Furthermore, this metric is easier to measure and relate to the firm's marketing efforts than other metrics of performance, e.g., sales, that depend on several other factors such as retail availability or income shocks.

A consumer in our model behaves according to a given *rule of thumb*; first, she treats all information she receives equally as opposed to putting different weights depending on when the information was received or on the identity of the sender. Moreover, when she engages in communication with her peers she passes a message regarding a brand with probability equal to her level of awareness. Although it would be interesting to extend the present model along these directions, we believe that our current formulation not only leads to a tractable analysis but it is also sufficient for the purposes of studying the effect of network structure on the outcomes of competition in targeted marketing.

3. Asymptotic behavior

We begin our analysis by characterizing the limiting behavior of the agents' brand awareness. First, we show that the vector of awareness levels converges almost surely and second we relate this limit to the properties of the underlying social network structure. Armed with this characterization, we study the problem that firms are facing: how to optimally allocate their advertising funds to the population of agents so as to maximize the average awareness in the society towards their brands.

We define the vector $\mathbf{y^A}[k] \in [0,1]^{(n+3)\times 1}$ such that $y_i^A[k] = x_i^A[k]$ for $i \in \mathcal{N}$, $y_{n+1}^A[k] = 1$ and $y_{n+2}^A[k] = y_{n+3}^A[k] = 0$. The first n entries of the vector $y^A[k]$ correspond to the awareness levels of the corresponding agents about brand A. The $(n+1)$-th and $(n+2)$-th entries correspond to firms A and B respectively, and the $(n+3)$-th entry corresponds to the status quo option. Similarly, we define $\mathbf{y^B}[k]$, where $y_i^B[k] = x_i^B[k]$ for $i \in \mathcal{N}$, $y_{n+1}^B[k] = y_{n+3}^B[k] = 0$, and $y_{n+2}^B[k] = 1$. In other words, to simplify the exposition of results we construct vectors $\mathbf{y^A}[k]$ and $\mathbf{y^B}[k]$ by adding three "dummy" agents corresponding to A, B, and SQ that have fixed awareness levels (always equal to 0 or 1).

Next we prove that both vectors of awareness levels converge almost surely to (deterministic) limits that can be characterized as a function of the network structure and the marketing allocations of the firms. Before stating the result, we define matrices W and V as:

$$[W]_{ij} = \begin{cases} -\frac{1}{n} & \text{if } i=j \text{ and } i \in \mathcal{N}, \\ \frac{1}{n}\alpha e_{ij} & \text{if } j \in \mathcal{N}_i \text{ and } i \in \mathcal{N}, \\ \frac{1}{n}(1-\alpha)\ h(b_i(A), b_i(B)) & \text{if } j = n+1 \text{ and } i \in \mathcal{N}, \\ \frac{1}{n}(1-\alpha)\ h(b_i(B), b_i(A)) & \text{if } j = n+2 \text{ and } i \in \mathcal{N}, \\ \frac{1}{n}(1-\alpha)\ (1-h(b_i(A), b_i(B)) - h(b_i(B), b_i(A))) & \text{if } j = n+3 \text{ and } i \in \mathcal{N}, \\ 0 & \textit{otherwise}. \end{cases} \tag{1}$$

and

$$[V]_{ij} = \begin{cases} \frac{[W]_{ij}}{-[W]_{ii}} & \text{if } j \neq i,\ [W]_{ii} \neq 0, \\ 0 & j \neq i,\ [W]_{ii} = 0, \\ 0 & j = i,\ [W]_{ii} \neq 0, \\ 1 & j = i,\ [W]_{ii} = 0. \end{cases}$$

Then, we obtain:

Proposition 1. *The awareness level vectors $y^A[k], y^B[k]$ converge almost surely to vectors y^A_{lim}, y^B_{lim} respectively. Moreover, the limiting vectors y^A_{lim} and y^B_{lim} are unique and satisfy:*

$$y^A_{lim} = \lim_{k\to\infty} V^k y^A[0],$$
$$y^B_{lim} = \lim_{k\to\infty} V^k y^B[0].$$

To gain intuition on Proposition 1, note that matrix V can be thought of as the transition matrix of a Markov chain with three absorbing states (corresponding to the two firms and the status quo, i.e., entries $n+1$, $n+2$, and $n+3$). Thus, matrix $\lim_{k\to\infty} V^k$ has non-zero elements only in columns $n+1$, $n+2$ and $n+3$. Moreover, it is straightforward to see that the i-th element of vector y^A_{lim} is equal to *the probability that the random walk defined by matrix V is absorbed by state $n+1$ (as opposed to being absorbed by states $n+2$ or $n+3$) when it is initiated at node i*. Similarly, the i-th element of vector y^B_{lim} is equal to the probability that the random walk defined by matrix V is absorbed by state $n+2$. Finally, since $\lim_{k\to\infty} V^k$ has non-zero elements only in the last three columns, the limiting awareness levels for the agents are independent of their values at time $k=0$, i.e., they do not depend on vectors $\mathbf{x^A}[0]$ and $\mathbf{x^B}[0]$.

In the rest of the section, we provide a characterization of the limiting value for the average awareness level for each of the two brands in the population of agents. In particular, let $m^A[k] = \frac{1}{n}\sum_{i=1}^n y^A_i[k]$, and $m^B[k] = \frac{1}{n}\sum_{i=1}^n y^B_i[k]$ be the average awareness levels at time period k. Then, it is straightforward to see that they converge almost surely to the unique values m^A_{lim}, m^B_{lim} defined as:

$$m^A_{lim} = \frac{1}{n}\sum_{i=1}^{n}[y^A_{lim}]_i,$$
$$m^B_{lim} = \frac{1}{n}\sum_{i=1}^{n}[y^B_{lim}]_i.$$

We can partition matrix V as:

$$V = \begin{bmatrix} F & E \\ 0 & I_3 \end{bmatrix}, \tag{2}$$

where I_3 is the 3×3 identity matrix, $F \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times 3}$. Matrix F captures the communication patterns among agents, i.e., it is equal to the adjacency matrix of the underlying social network $\mathcal{E}$ scaled by α. Moreover, $[E]_{i1} = (1-\alpha)\ h(b_i(A), b_i(B))$ and $[E]_{i2} = (1-\alpha)\ h(b_i(B), b_i(A))$, for all $i \in \mathcal{N}$. Given this partitioning, and denoting $[.]_i$ as the i-th column of its argument, we obtain the following theorem:

Theorem 1. *The average awareness levels in the population of agents converge almost surely to:*

$$m^A_{lim} = \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)[\mathbf{1}'(I-F)^{-1}]_i\ h(b_i(A), b_i(B)),$$
$$m^B_{lim} = \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)[\mathbf{1}'(I-F)^{-1}]_i\ h(b_i(B), b_i(A)).$$

Theorem 1 provides a sharp characterization of the average awareness levels at the limit as a function of the underlying network structure and the allocations chosen by the two firms. Specifically, quantity $c_i = [\mathbf{1}'(I-F)^{-1}]_i$ defines a *centrality* measure for node i. In particular, c_i is equal to the expected number of visits to node i before absorption at either node $n+1$, $n+2$, or $n+3$ for a random walk with transition probability matrix V that was originated at a node other than i uniformly at random. Thus, Theorem 1 states that *the average awareness levels at the limit are equal to a weighted sum of the firms' marketing efforts where the weights are given by the centralities of the agents.*

This notion of centrality is based on the concept that connections to agents with high centrality contribute more to the centrality of an agent than connections to agents with low centrality. Therefore, calculating an agent's centrality index takes the entire network structure into account, as opposed, for example, to *degree centrality* that is defined as the number of links incident to an agent and requires only local information. Moreover, Theorem 1 captures the fact that when firms devise their optimal marketing strategies they should account for *multiplier effects*: the return to their marketing strategies depends not only on the agents they directly target but also on how well connected the latter are with other central agents.

4. Optimal budget allocation

In this section we turn our attention to the optimization problem that the two competing firms are facing, i.e., how to optimally allocate their advertising funds to the population of agents. First, we characterize the best response of a firm to a fixed budget allocation decision of its competitor. In particular, assuming that firm B has already chosen to allocate its advertising funds according to vector $\{b_i(B)\}_{i\in\mathcal{N}}$ we provide an expression for the optimal budget allocation for firm A, i.e., the budget allocation that maximizes m^A_{lim}. Section 5 builds on the results of the present section and studies the equilibria of the competition between the two firms.

Given the total budget of firm A, i.e., C, and the budget allocation decisions of firm B, $\{b_i(B)\}_{i\in\mathcal{N}}$, firm A's optimization problem can be written as follows:

$$\begin{aligned} \max_{\{b_i(A)\}_{i\in\mathcal{N}}} \ & \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)c_i\ h(b_i(A),b_i(B)), \\ & \text{s.t.,} \\ & \sum_{i\in\mathcal{N}} b_i(A)\le C, \\ & b_i(A)\ge 0, \text{ for all } i\in\mathcal{N}. \end{aligned} \tag{3}$$

The following lemma characterizes the optimal allocation for firm A as a function of the allocation chosen by firm B.

Lemma 1. *If $\{b^\star_i(A)\}_i\in\mathcal{N}$ is a solution to optimization problem (3), then there exist $\{\lambda_i\}_{i\in\mathcal{N}}$ and γ such that:*

$$\begin{array}{cl} b^\star_i(A)\ge 0 & \text{for all } i\in\mathcal{N}, \\ \sum_{i\in\mathcal{N}} b^\star_i(A)\le C & \\ \lambda_i\ge 0 & \text{for all } i\in\mathcal{N}, \\ \lambda_i b^\star_i(A)=0 & \text{for all } i\in\mathcal{N}, \\ -\frac{1}{n}c_i(1-\alpha)\ \left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b^\star_i(A)} +\gamma-\lambda_i=0 & \text{for all } i\in\mathcal{N}. \end{array}$$

Finally, if $h(x,y)$ is strictly concave in x for all $y\in[0,C]$, the solution to firm A's problem is unique and it is given by the proposition below.

Proposition 2. *Suppose that $h(x,y)$ is strictly concave in x for all $y\in[0,C]$. Then, optimization problem (3) has a unique solution $\{b^\star_i(A)\}_i\in\mathcal{N}$ which satisfies:*

$$b^\star_i(A)=\begin{cases} 0 & \text{if } \gamma\ge \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \\ b_i(A)\ \text{s.t.}\ \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{b=b_i(A)}=\gamma & \text{if } \gamma< \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \end{cases}$$

and $\sum_{i\in\mathcal{N}} b^\star_i(A)=C$.

Proposition 2 essentially states that firm A allocates positive advertising budget on an individual i if she is *central* enough in the social network structure, as implied by her centrality index c_i. This observation implies the following proposition, i.e., advertising funds targeted to an individual are increasing with the individual's centrality.

PROPOSITION 3. *Suppose that $h(x,y)$ is strictly increasing in x for all $y \in [0,C]$, and that agents $i,j \in \mathcal{N}$ are such that $b_i(B) = b_j(B)$ and $c_i \geq c_j$. Then, $b_i^\star(A) \geq b_j^\star(A)$.*

5. Equilibria of the competition game

In this section, we study the equilibria of the marketing competition between firms A and B. First, we describe the game:

Allocating the advertising funds: Firms A and B choose how to allocate their advertising budgets to individuals, so as to maximize their brand awareness levels, i.e., they choose $b^A = [\{b_i(A)\}_{i\in\mathcal{N}}]$ and $b^B = [\{b_i(B)\}_{i\in\mathcal{N}}]$ in order to maximize m_{lim}^A, m_{lim}^B respectively.
Information exchange: Consumers obtain information from their peers and the competing firms according to the process outlined in previous sections.

Let $\mathcal{B}_A$ denote the set of pure strategies of firm A, i.e., $b^A \in \mathcal{B}_A$. Similarly, we let $\mathcal{B}_B$ denote the set of pure strategies of firm B. With a slight abuse of notation, we let $m_{lim}^A(b^A, b^B)$ denote the average awareness level at the limit for brand A, m_{lim}^A, when A employs strategy b^A and B employs strategy b^B. We assume that the firms have the same adverting budget denoted by C. We impose no other restrictions on $\mathcal{B}_A$ and $\mathcal{B}_B$. Therefore, $\mathcal{B}_A = \{b^A \in \mathbb{R}^n \big| b^A \geq 0, \sum_{i\in\mathcal{N}} b_i(A) \leq C\}$ and $\mathcal{B}_B = \{b^B \in \mathbb{R}^n \big| b^B \geq 0, \sum_{i\in\mathcal{N}} b_i(B) \leq C\}$.

DEFINITION 1. A pair of pure strategies $(b^A, b^B), b^A \in \mathcal{B}_A, b^B \in \mathcal{B}_B$, is a Nash equilibrium for the marketing competition game defined above if and only if:

$$m_{lim}^A(b^A, b^B) \geq m_{lim}^A(\tilde{b}^A, b^B) \text{ for all } \tilde{b}^A \in \mathcal{B}_A,$$
$$m_{lim}^B(b^A, b^B) \geq m_{lim}^B(b^A, \tilde{b}^B) \text{ for all } \tilde{b}^B \in \mathcal{B}_B.$$

We show that the marketing competition game has a Nash equilibrium in pure strategies.

PROPOSITION 4. *Suppose that function $h(x,y)$ is strictly increasing in x for every $y \in [0,C]$. Then, the marketing competition game has a pure strategy Nash equilibrium.*

In the remainder of the section, we characterize the equilibria of the marketing competition game under different conditions on function h. We show that a symmetric equilibrium always exists and, more interestingly, we provide conditions on h and the network structure under which asymmetric equilibria also exist. We restrict attention to h that is strictly concave in x which corresponds to the case of diminishing returns to advertising, a quite natural assumption in our setting.

As a way of previewing the results in this section, we discuss the main effects (along with the network) that determine the structure of equilibrium allocations in the game:

(1) Diminishing returns to advertising: As mentioned above, it is quite natural to assume that the returns to advertising are diminishing. This essentially puts an upper bound on the marketing budget a firm is willing to spend on an individual and creates the incentive for the firms to spread their marketing efforts across the population of agents which in turn leads to more similar allocations from the two firms.

(2) Strategic substitutability or complementarity of advertising: The two firms are competing both between each other and with the status quo option, i.e., the incumbent firm or the decay in aggregate awareness. Their efforts are strategic complements or substitutes depending on whether an increase on a firm's advertising efforts directed to an individual increases or decreases the marginal returns to advertising for its competitor.[9]

Case 1: Concave h, advertising efforts are complements

First, we consider the environment defined by Assumption 3 below.

Assumption 3. *Function h is strictly increasing and strictly concave in x and the mixed partial derivative is non-negative for all $x,y \in [0,C]$, i.e.:*

$$\frac{\partial h(x,y)}{\partial x} > 0, \ \frac{\partial^2 h(x,y)}{\partial x^2} < 0, \ \frac{\partial^2 h(x,y)}{\partial x \partial y} \geq 0.$$

We show that all equilibria of the marketing competition game under Assumption 3 are *symmetric*, i.e., the advertising budget that firm A allocates to individual i is the same as the one that B allocates to i for all individuals.

Proposition 5. *All Nash equilibria of the marketing competition game under Assumption 3 are symmetric, i.e., if b^A, b^B is an equilibrium allocation, then $b_i(A) = b_i(B)$ for all $i \in \mathcal{N}$.*

Assumption 3 describes an environment where a firm's advertising effort exerts a positive externality on its competitor as the mixed partial derivative is non-negative, i.e., the firms' actions are strategic complements. It is then straightforward to show that symmetric equilibria exist and, moreover, no asymmetric allocations, i.e., allocations where there exists at least an agent i for which $b_i(A) \neq b_i(B)$, can be at equilibrium.

[9] For a thorough discussion on strategic complementarity and substitutability in the context of multi market oligopoly competition refer to Bulow et al. (1985).

Case 2: Concave h, advertising efforts are not complements

The second case involves an environment where the dynamics of information exchange are governed by a strictly concave h, which is such that advertising efforts are not complements, i.e., $\frac{\partial^2 h(x,y)}{\partial x \partial y} < 0$. Then, increasing a firm's budget allocation on an individual exerts a negative externality on its competitor. We further distinguish between two subcases depending on the relative sizes of $\left|\frac{\partial^2 h}{\partial x \partial y}\right|$ and $\left|\frac{\partial^2 h}{\partial x^2}\right|$. We let $\mathcal{A}_A$ and $\mathcal{A}_B$ denote the *activation sets* of firms A and B respectively, i.e., the sets of agents that receive a positive fraction of the advertising budget from the respective firm at equilibrium.

Proposition 6. *Suppose that function h is strictly increasing and strictly concave in x and $\frac{\partial^2 h(x,y)}{\partial x \partial y} < 0$ for all $x, y \in [0, C]$. Moreover, h satisfies the following:*

$$\left|\frac{\partial^2 h(x_1, x_2)}{\partial x^2}\right| > \left|\frac{\partial^2 h(x_2, x_1)}{\partial x \partial y}\right|, \text{ for all } 0 \leq x_2 < x_1 \leq C. \tag{4}$$

Then, the marketing competition game has a unique Nash equilibrium which is symmetric.

This result can be understood in terms of the effects identified above as follows: the competition between the two firms, which is captured by $\left|\frac{\partial^2 h(x,y)}{\partial x \partial y}\right|$, is not sufficiently intense when (4) is satisfied, and therefore firms find it optimal to spread their budgets across the population of agents (the diminishing returns effect dominates the strategic substitutability of advertising efforts). Thus, the equilibrium ends up being unique and symmetric.

The conditions in Propositions 5 and 6 that preclude the existence of asymmetric equilibria are quite strong. In particular, they imply that asymmetric equilibria do not exist when either the firms' advertising efforts are complements (because the rate of decay in their aggregate awareness is high) or when the rate at which the returns to targeting are diminishing is high (and the firms have a strong incentive to spread their marketing budgets). Deviating from these conditions makes the existence of asymmetric equilibria possible. The following proposition provides *sufficient* (but not necessary) conditions[10] that guarantee their existence. They essentially require that the competition between the two firms is sufficiently intense, i.e., a firm's advertising efforts directed to an individual negate the efforts of its competitor and strategic substitutability dominates diminishing returns, and there are more than one agents that are worth targeting.

Proposition 7. *Suppose that function h is strictly increasing and strictly concave in x and $\frac{\partial^2 h(x,y)}{\partial x \partial y} < 0$ for all $x, y \in [0, C]$. Moreover, h satisfies the following:*

$$\left|\frac{\partial^2 h(x_1, x_2)}{\partial x^2}\right| < \left|\frac{\partial^2 h(x_2, x_1)}{\partial x \partial y}\right|, \text{ for all } 0 \leq x_2 < x_1 \leq C. \tag{5}$$

[10] Asymmetric equilibria exist even when Condition (5) does not hold for all x_1, x_2 such that $0 \leq x_2 < x_1 \leq C$.

Then, the marketing competition game has a symmetric Nash equilibrium. Moreover, if there are at least two agents k,ℓ in the activation sets for the symmetric equilibrium, i.e., $|\mathcal{A}_A| = |\mathcal{A}_B| \geq 2$, then there exist asymmetric equilibria.

The main qualitative take-away message of Proposition 7 is that asymmetric equilibria arise in a symmetric environment when competition between the two firms is sufficiently intense (as indicated by equation (5)) and there exists at least a pair of agents whose centralities are sufficiently high (so that they are worth targeting).

We conclude this section by explicitly constructing symmetric and asymmetric equilibria for different network structures.

Examples

Motivated by the discussion in Goyal and Kearns (2012), for the examples that follow we consider functions h that take the following form: $h(x,y) = f(x,y)g(x,y)$ (functions f and g correspond to the *switching* and *selection* functions in Goyal and Kearns (2012)). More specifically, we let

$$f(x,y) = \left[\frac{x+y}{x+y+\beta}\right]^r \text{ and } g(x,y) = \frac{x^s}{x^s + y^s}.$$

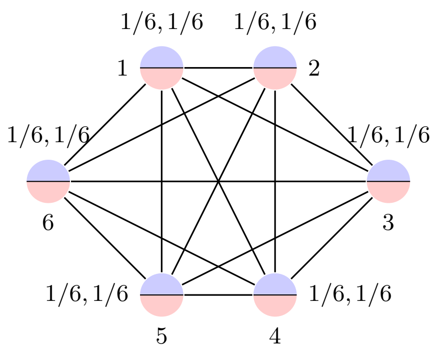
Parameter β captures the decay in the aggregate awareness for the two brands due to the forgetting effect (or alternatively can be thought of as the marketing efforts of the status quo option). Function f captures both the positive effect that a competitor's marketing efforts has on a firm's awareness level ("confusion effect" (Pauwels (2004))) as well as the decay in aggregate awareness (constant β). On the other hand, g models the competition between the two firms. The form for function g is known as the *Tullock contest function* (Tullock (1980)). When $s = 1$ both firms are treated equally, whereas for $s < 1$ the firm that is allocating a lower budget is favored relative to the case when $s = 1$ and for $s > 1$ the firm that is allocating a higher budget is favored. This intuitively implies that for higher values of s the competition between the two firms is softened. Indeed, when $r = s = 1$ there are no asymmetric equilibria whereas for $s > 1$ asymmetric equilibria exist as seen below. Specifically, consider the following two choices for h:

$$h(x,y) = \left[\frac{x+y}{x+y+\beta}\right] \cdot \frac{x}{x+y} = \frac{x}{x+y+\beta}$$

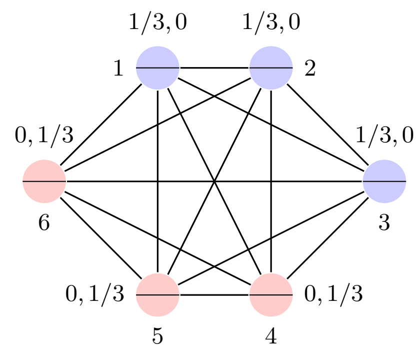
and

$$h(x,y) = \left[\frac{x+y}{x+y+\beta}\right] \cdot \frac{x^2}{x^2+y^2}.$$

Then, a symmetric equilibrium exists for both. More interestingly, there exist asymmetric equilibria only for the second one. Below, we describe symmetric and asymmetric equilibria for the following two network structures when C is normalized to 1: the complete network with 6



(a) Symmetric equilibrium.



(b) Asymmetric equilibrium: The firms split the nodes.

Figure 1 A symmetric and an asymmetric equilibrium for the complete network.

agents and a barbell network with 25 agents.

Complete network

In the complete network, all agents have the same centrality, i.e., $c_i = c_j$ for all $i, j \in \mathcal{N}$. Moreover, let $\beta = 0.5$. The following are equilibria of the marketing competition game:

(a) Symmetric: There is a symmetric equilibrium in this example, where both firms allocate 1/6 of their budget to each of the agents, i.e., $b_i(A) = b_i(B) = 1/6$ for all $i \in \mathcal{N}$.

(b) Asymmetric: The following allocation in which the two firms do not allocate advertising funds to the same individual is an asymmetric equilibrium of the marketing competition game:

$$b_1(A) = b_2(A) = b_3(A) = b_4(B) = b_5(B) = b_6(B) = 1/3,$$

and

$$b_1(B) = b_2(B) = b_3(B) = b_4(A) = b_5(A) = b_6(A) = 0.$$

Barbell network

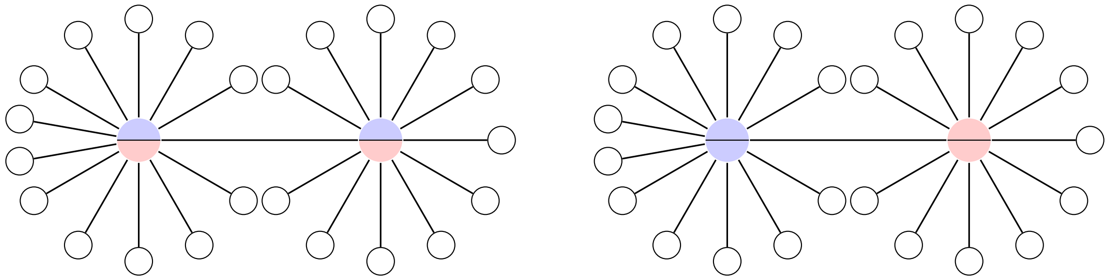
Consider a barbell network with 25 nodes. The network consists of two subgraphs: a 13 node star network and a 12 node star network. Subnetworks are connected to each other through the star nodes. For $\alpha = 0.5$, centralities are given by:

$$c_1 = 9.58, \quad c_2 = 8.91, \quad c_3 = \cdots = c_{14} = 1.3685, \quad c_{15} = \cdots = c_{25} = 1.3712.$$

As before we let $\beta = 0.5$. The following are equilibria of the marketing competition game:

(a) Symmetric:

$$b[A] = b[B] = [0.5238,\ 0.4762,\ 0,\ \cdots,\ 0]^T.$$



(a) Symmetric equilibrium.

(b) Asymmetric equilibrium: The firms split the two star nodes.

Figure 2 The symmetric and one asymmetric equilibrium for a barbell network.

(b) Asymmetric:

$$b[A] = [1,\ 0,\ 0,\ \cdots,\ 0]^T,$$
$$b[B] = [0,\ 1,\ 0,\ \cdots,\ 0]^T.$$

6. When is targeting most useful?

A question that arises naturally is whether targeting leads to a significant increase in the success of a marketing campaign, and if so, how does the latter depend on the properties of the underlying social network structure. To provide intuition on this, we characterize the average awareness level at the limit for firm A when A optimally allocates its advertising budget to the population of agents, whereas firm B distributes its budget uniformly to the entire population of agents. To obtain closed-form analytical expressions for the benefits of targeted marketing, we assume that $h(x,y) = \frac{x}{x+y+\beta}$, where recall that β captures the intensity of the aggregate awareness decay effect (or the marketing efforts of the status quo option). Moreover, we consider networks for which it is optimal for both firms to allocate a positive fraction of their marketing budgets to all agents (this is imposed by condition (7) below). In some sense this makes for a more fair comparison as firm B does not waste any of its budget on agents that firm A finds optimal to completely ignore.

Informally, Theorem 2 states that targeting is most useful when there is a lot of heterogeneity among the agents in terms of their *network influence* captured by their respective centrality.

THEOREM 2. *Suppose that firm B is allocating its marketing budget, C, uniformly across all agents, i.e.,* $b_i(B) = \frac{C}{n}$*, for all* $i \in \mathcal{N}$*. Then, the average awareness level in the population of agents for firm A, when firm A allocates its marketing budget optimally, is given by*

$$m^A_{lim} = 1 - \frac{1-\alpha}{n^2}\frac{C+n\beta}{2C+n\beta}\left(\sum_{i\in\mathcal{N}}\sqrt{c_i}\right)^2, \tag{6}$$

when

$$\frac{\sqrt{c_i}(2C+n\beta)}{\sum_{i\in\mathcal{N}}\sqrt{c_i}}-\frac{C}{n}-\beta\geq 0. \tag{7}$$

Condition (7) essentially implies that the differences in the agents' centralities are not too large and thus it is optimal for firm A to allocate a fraction of its marketing budget to every agent. Theorem 2 provides a concrete way for a firm to assess the return in targeting its marketing resources. This gain is larger when there are highly central agents, i.e., agents whose centrality is much higher than the average. In particular, the returns to targeting are maximized in *star-like* networks, where a small subset of agents are directly connected to everyone else. On the other hand, if agents' centralities do not differ by much, i.e., the network is symmetric, then targeting does not offer a competitive advantage. To see this, note that the average awareness level in the population of agents for firm A is proportional to the following quantity $\left(\sum_{i\in\mathcal{N}}\sqrt{c_i}\right)^2$. Moreover, note that by definition, the sum of the centrality indices is fixed and equal to $\frac{n}{1-\alpha}$, i.e., $\sum_{i\in\mathcal{N}}c_i=\frac{n}{1-\alpha}$. Thus, the average awareness level m^A_{lim} is minimized when all agents have the same centrality (complete network), whereas it is maximized when there is an agent with a much higher centrality than the rest (star-like network) as is also illustrated by the examples below.

Examples

Complete network: When the network is complete, the centralities of all agents are equal to $c_i=\frac{1}{1-\alpha}$, for all $i\in\mathcal{N}$. Thus, we obtain from equation (6):

$$m^{A,complete}_{lim}=\frac{C}{2C+n\beta}.$$

If $C=n\beta$, i.e., the three options (firms) are symmetric in terms of their aggregate budget, we obtain that $m^{A,complete}_{lim}=1/3$. Firm A can only capture 1/3 of the target market, i.e., the same fraction as firm B, and targeted marketing does not offer any advantage.

Star-like network: We focus on a network with a star-like structure, where $c_1>c_2=\cdots=c_n$. Moreover, assume that c_i's are such that the activation set includes all nodes, i.e., it is optimal for firm A to allocate non-zero marketing budget to all nodes. For this to be true, the following should hold,

$$c_1\frac{b_1(B)+\beta}{(b_1(A)+b_1(B)+\beta)^2}\geq c_i\frac{b_i(B)+\beta}{(b_i(A)+b_i(B)+\beta)^2},\ \text{for all } i\in\mathcal{N}/\{1\}.$$

Thus, the maximum value that c_1 can take when $b_i(B)=C/n$ for all $i\in\mathcal{N}$ is given by

$$c_1\frac{C+n\beta}{(nC+C+n\beta)^2}=c_2\frac{C+n\beta}{(C+n\beta)^2},$$

and $c_1+(n-1)c_2=\frac{n}{1-\alpha}$. From these two expressions we obtain,

$$c_1=\frac{n}{1-\alpha}\frac{Z^2}{Z^2+(n-1)}\ \text{and}\ c_i=\frac{n}{1-\alpha}\frac{1}{Z^2+(n-1)},\ \text{for all } i\in\mathcal{N},$$

with $Z = \frac{nC+C+n\beta}{C+n\beta}$. Finally,

$$m_{lim}^{A,star} = 1 - \frac{1}{n}\frac{C+n\beta}{2C+n\beta}\frac{(Z+(n-1))^2}{Z^2+(n-1)}. \tag{8}$$

If $C = n\beta$, i.e., the three options (firms) are symmetric in terms of their aggregate budget we obtain that:

$$m_{lim}^{A,star} \to 1, \text{ as } n \to \infty,$$

i.e., firm A captures the entire target market as the number of customers grows to infinity by appropriately targeting its advertising budget.

7. How does the network structure affect competition?

Our final set of results explores the question of how the network structure affects the outcomes of targeting under competition. In contrast to the previous section, we assume that both firms make use of targeting technologies and examine the ratio of their levels of brand awareness as a function of the network structure at equilibrium. Our goal is to provide intuition on what networks may lead to a large such ratio, i.e., a large difference in the outcomes of the marketing campaigns of the two firms. Specifically, we are interested in the following quantity:

$$\Pi = \max_{\mathcal{E},(\sigma^A,\sigma^B)} \frac{m_{lim}^A(\sigma^A,\sigma^B)}{m_{lim}^B(\sigma^A,\sigma^B)},$$

where $\sigma = (\sigma^A, \sigma^B)$ is an equilibrium of the marketing competition game when the adjacency matrix corresponding to the social network structure is given by $\mathcal{E}$. Let $c_1, c_2, \cdots, c_n$ denote the centralities of the agents under $\mathcal{E}$ and, without loss of generality, assume that $c_1 \geq c_2 \geq \cdots \geq c_n$ (recall that $c_1 + \cdots + c_n = \frac{n}{1-\alpha}$). Finally, for the rest of the section, we focus our attention to functions $h(.)$ that satisfy Assumption 4 below.

Assumption 4. *Let $h(x,y)$ be strictly increasing in x, strictly concave in x, and such that*

$$\frac{\partial^2 h(x_2,x_1)}{\partial x \partial y} < \frac{\partial^2 h(x_1,x_2)}{\partial x^2} < 0, \text{ for all } 0 \leq x_2 < x_1 \leq C.$$

Additionally, assume that $h'(0,0) < \epsilon$ for some ϵ.

Proposition 7 implies that asymmetric equilibria exist under Assumption 4. The following theorem provides a lower bound on ratio Π.[11]

Theorem 3. *Suppose that h satisfies Assumption 4. Then, there exists $\bar{n}$ such that when the society has $n \geq \bar{n}$ agents, the ratio Π of awareness levels is bounded below by:*

$$\Pi \geq \frac{h'(C,0)}{h'(0,C)}.$$

[11] The upper bound on $h'(0,0)$ is not crucial for the qualitative insights generated by Proposition 3, however it greatly simplifies the presentation of the result.

At this point, it would be instructive to describe the main idea behind the lower bound in Theorem 3. An asymmetric equilibrium leads to large differences in the awareness levels for the two firms when one allocates its budget to agents with higher centralities than the other. If it is optimal for the firms to target only the two agents with highest centrality then Π simply satisfies: $\Pi \geq \frac{c_1 h(C,0)}{c_2 h(C,0} = c_1/c_2$. However, for such an allocation to be an equilibrium the difference between the centralities c_1, c_2 cannot be too high, as one of the firms would have an incentive to deviate. More specifically, from the results in Section 4 this allocation is an equilibrium if the following relations hold:

$$c_1 h'(0,C) \leq c_2 h'(C,0) \text{ and } c_2 h'(C,0) \geq c_3 h'(0,0).$$

The lower bound is obtained when the first inequality holds with equality. To complete the proof of Theorem 3 we explicitly construct a network structure for which this allocation is an equilibrium of the marketing competition game.

The take-away message of Theorem 3 is that network structures that lead to unequal equilibrium outcomes generally feature two groups of agents with different but sufficiently high centralities. In that case, there exists an asymmetric equilibrium where each of the two firms targets one of the groups and, thus, the ratio between their awareness levels is maximized. As in the previous section, heterogeneity in the underlying network structure leads to higher benefits from targeting for the firm that allocates its budget to an appropriate group of individuals. However, when both firms have access to targeting technologies what increases the benefits from targeting is heterogeneity among the agents that have *sufficiently high centrality* and the networks that lead to large differences in the outcomes of marketing campaigns have a barbell-like structure.

8. Heterogeneous agents

It is most often the case that consumers are heterogeneous with respect to their preferences and, as a result, marketing efforts may lead to substantially different behavior depending on the consumer segment they are targeted to. Thus, when devising a marketing campaign, a firm has to take into consideration the characteristics of the consumer population it is targeting. In this section, we describe how to incorporate heterogeneity in terms of the agents' preferences in the model and discuss how this additional layer of complexity affects our results.

A simple way to introduce heterogeneity is by assuming that each consumer i has a bias towards firm A or B given by parameters r_i^A and r_i^B respectively. The latter capture the probability with which the consumer updates her awareness level upon receiving a message in favor of firm A or B. Specifically, consumer i updates her awareness level in favor of firm A with probability r_i^A upon receiving a positive message for firm A, and similarly for firm B. Otherwise, the agent's awareness level remains unchanged. Thus, the lower r_i^A is the more likely it is that individual i will ignore firm A's advertising.[12]

The qualitative nature of our results remains essentially the same since even in the extended framework firms choose to target central individuals. However, the average awareness levels incorporate the agents' preferences captured by vectors $\{r_i^A\}_{i\in\mathcal{N}}$ and $\{r_i^B\}_{i\in\mathcal{N}}$. Specifically, the average awareness level for firm A is given by:

$$m_{lim}^A = \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)\ r_i^A\ [\mathbf{1}'(I-F)^{-1}]_i\ h(b_i(A), b_i(B)). \tag{9}$$

The heterogeneity in preferences does not affect the information exchange process among the agents, therefore the structure of matrix $(I-F)$ remains unchanged. However, note that the centrality of agent i is scaled by r_i^A in Expression (9). Therefore, firm A's budget allocation to agent i increases not only with i's centrality but also with her preference towards A. Finally, asymmetric equilibria may now arise even when the conditions in Propositions 5 and 6 are satisfied.

9. Conclusions

The paper studies a stylized model of marketing competition between two firms over social networks. We provide a concrete characterization of the optimal marketing strategies for the two competitors and clearly illustrate that network centrality is an important metric of an agent's influence over her peers. Furthermore, we highlight the value of targeted advertising and its dependence on the structure of the underlying network. Finally, we show that equilibrium behavior over networks may lead to large differences in the levels of brand awareness for the two firms even when their marketing budgets are the same.

Although agents in our model behave according to a simple rule-of-thumb (for example, they put an equal weight to every piece of information they receive), we believe that our model captures many of the essential features of targeted marketing and its impact on the evolution of brand awareness: advertising efforts have typically diminishing returns, a competitor's advertising may have both positive and negative effects on a firm's own awareness level, and awareness may decay over time due to "forgetting".

Our main focus in this paper is to study the word-of-mouth process and how competing firms can influence it. Thus, we chose to use the firm's level of brand awareness as its objective. Extending our analysis to incorporate purchasing decisions from the agents and pricing decisions from the firms is an interesting direction for future research. In particular, consider a setting where consumers choose

[12] Obviously, there are several other ways to introduce heterogeneity in the model. However, the point of this subsection is not to fully explore heterogeneity but rather to illustrate that our results remain largely unaffected (at least qualitatively).

whether and when to purchase one of the competing products depending on their beliefs about their (originally unknown) quality and the respective prices. Furthermore, consumers can generate and pass information to their peers from their own experimentation with the product. Firms in this richer framework can control the rate of learning both through informative advertising and their pricing decisions. What would be then the optimal combined pricing and marketing strategies for the two competitors? Does targeted advertising or targeted pricing lead to higher profits?[13] Finally, marketing budgets were assumed to be fixed and exogenously determined. A reasonable extension would be to consider endogenizing the choice of a firm's marketing budget and derive its optimal level as a function of the competition and the consumers' network structure.

On a broader level, we feel that most of the literature related to social networks so far has focused on providing models on how individuals interact with their peer group and at what extent the latter influences the decisions taken by the former. However, we believe that there is an urgent need for research that uses our current understanding of this interaction as a starting point and prescribes ways to improve on outcomes. We view the present work as a step towards this direction and we certainly find that there are many more interesting research avenues to explore.

Appendix

Proof of Proposition 1

The proof of Proposition 1 is based on Lemmas 2 and 3. Lemma 2 characterizes the expectation of awareness levels for brands A and B at time k as a function of the history up to (and excluding) period k.

LEMMA 2. *Given a society $\mathcal{G}(\mathcal{N},\mathcal{E})$, budget allocation decisions $\{b_i(A), b_i(B)\}_{i\in\mathcal{N}}$, and the history of information exchange up to time period k, $\{r[l], s[l], m[l]\}_{l<k}$, the expected value of the agents' awareness levels at time k is given by:*

$$\mathbb{E}[y^A[k] \big| \{r[l], s[l], m[l]\}_{l<k}] = (I + D[k-1]W)\, y^A[k-1],$$
$$\mathbb{E}[y^B[k] \big| \{r[l], s[l], m[l]\}_{l<k}] = (I + D[k-1]W)\, y^B[k-1],$$

where I is the$(n+3)\times(n+3)$ identity matrix, W is given in Equation (1), and $D[k-1]$ is a diagonal matrix with:

$$[D[k-1]]_{ii} = \begin{cases} (|\{l : r[l] = i, l < k\}| + 1)^{-1} & \text{if } i \in \mathcal{N}, \\ 1 & \text{if } i \in \{n+1, n+2, n+3\}. \end{cases}$$

[13] For related work, refer to Bergemann and Valimaki (1996), Bergemann and Valimaki (2000), Iyer et al. (2005), Acemoglu et al. (2012), and Ifrach et al. (2013).

Proof. First, we study the dynamics of an agent's awareness level about firm A. Note that given the history of message exchanges $\{r[l], s[l], m[l]\}_{l<k}$, the awareness level of agent i at iteration $k-1$ is deterministic and equal to $x_i^A[k-1]$. For a given agent $i \in \mathcal{N}$ and iteration k:

$$\begin{aligned}\mathbb{E}[x_i^A[k] | \{r[l], s[l], m[l]\}_{l<k}] &= \mathbb{P}(r[k] \neq i)\ x_i^A[k-1] + \mathbb{P}(r[k] = i)\mathbb{E}[x_i^A[k] | \{r[l], s[l], m[l]\}_{l<k}, r[k] = i] \\ &= \frac{n-1}{n} x_i^A[k-1] + \frac{1}{n}\mathbb{E}[x_i^A[k] | \{r[l], s[l], m[l]\}_{l<k}, r[k] = i], \end{aligned} \tag{10}$$

where Equation (10) follows from the fact that the agent that receives a message at time k is chosen uniformly at random. The second term in Equation (10) is equal to:

$$\begin{aligned}\mathbb{E}[x_i^A[k] | \{r[l], s[l], m[l]\}_{l<k}, r[k] = i] = \alpha\mathbb{E}[x_i^A[k] | s[k] \in \mathcal{N}_i, \{r[l], s[l], m[l]\}_{l<k}, r[k] = i] \\ + (1-\alpha)\mathbb{E}[x_i^A[k] | s[k] \notin \mathcal{N}_i, \{r[l], s[l], m[l]\}_{l<k}, r[k] = i], \end{aligned} \tag{11}$$

where Equation (11) follows from the fact that if agent i is the receiver of the message at iteration k, then the sender of the message belongs to her peer group with probability α and with probability $(1-\alpha)$ the message comes directly from A, B, or SQ.

The first term of Equation (11) can be rewritten as:

$$\begin{aligned}&\mathbb{E}[x_i^A[k] \big| s[k] \in \mathcal{N}_i, \{r[l], s[l], m[l]\}_{l<k}, r[k] = i] = \\ &\quad = \frac{|\{l : r[l] = i, l \leq k\}|}{|\{l : r[l] = i, l \leq k\}| + 1} x_i^A[k-1] + \sum_{j \in \mathcal{N}_i} (\mathbb{P}(s[k] = j)\ x_j^A[k-1] \frac{1}{|\{l : r[l] = i, l \leq k\}| + 1} \\ &\quad = \frac{|\{l : r[l] = i, l \leq k\}|}{|\{l : r[l] = i, l \leq k\}| + 1} x_i^A[k-1] + \sum_{j \in \mathcal{N}_i} e_{ij}\ x_j^A[k-1]\ \frac{1}{|\{l : r[l] = i, l \leq k\}| + 1}\end{aligned}$$

The second term of Equation (11) can be rewritten as:

$$\begin{aligned}&\mathbb{E}[x_i^A[k] \big| s[k] \notin \mathcal{N}_i, \{r[l], s[l], m[l]\}_{l<k}, r[k] = i] = \\ &\quad = \frac{|\{l : r[l] = i, l \leq k\}|}{|\{l : r[l] = i, l \leq k\}| + 1} x_i^A[k-1] + h(b_i(A), b_i(B)) \frac{1}{|\{l : r[l] = i, l \leq k\}| + 1}\end{aligned}$$

Similarly, we obtain an expression for $\mathbb{E}[x_i^B[k] | \{r[l], s[l], m[l]\}_{l<k}]$. The lemma follows from simple algebra. ■

For the rest of the proof, we only study $y^A[k]$ (the analysis for $y^B[k]$ is identical). The next lemma describes the evolution of the awareness level vector from one time period to the next.

LEMMA 3. *The awareness level vector satisfies:*

$$y^A[k] = y^A[k-1] + D[k-1]\left(Wy^A[k-1] + n[k]\right). \tag{12}$$

Moreover, if $\{\mathcal{F}[l]\}_{l\geq 0}$ *is the increasing family of* σ *fields with*

$$\mathcal{F}[l] = \sigma(\{r[l], s[l], m[l]\}_{l<k}), \quad k \geq 0,$$

then $\mathbb{E}[n[k]|\mathcal{F}[k-1]] = 0$ *almost surely (a.s.), and* $\mathbb{E}[||n[k]||^2|\mathcal{F}[k-1] \leq \sup_{l<k} K(1+||y[l]||^2)$ *a.s., for all* $k \geq 1$*, and some constant* $K > 0$*.*

Proof. Expression (12) follows from Lemma 2 for some function $n[k]$. Next, note that $D[k-1]|\mathcal{F}[k-1]$ is invertible a.s. for all $k \geq 1$, since it is a diagonal matrix with non-zero entries a.s.. Then, we can rewrite Equation (12) as:

$$n[k] = D^{-1}[k-1]\left(y^A[k] - (I + D[k-1]W)y^A[k-1]\right).$$

By taking expectations on both sides conditional on the σ-field $\mathcal{F}[k-1]$ we have:

$$\begin{aligned}
\mathbb{E}[n[k]|\mathcal{F}[k-1]] &= \mathbb{E}[D^{-1}[k-1]\left(y[k] - (I + D[k-1]W)y^A[k-1]\right)|\mathcal{F}[k-1]] \\
&= \mathbb{E}[D^{-1}[k-1]y^A[k]|\mathcal{F}[k-1]] - \mathbb{E}[D^{-1}[k-1]\ (I + D[k-1]W)y^A[k-1]|\mathcal{F}[k-1]] \\
&= \mathbb{E}[D^{-1}[k-1]|\mathcal{F}[k-1]]\mathbb{E}[y^A[k]|\mathcal{F}[k-1]] - \mathbb{E}[(D^{-1}[k-1] + W)y^A[k-1]|\mathcal{F}[k-1]] = 0,
\end{aligned}$$

where the last equality follows from Lemma 2. For a given history of message exchanges $\{r[l], s[l], m[l]\}_{l<k}$, vector $y_i^A[k]$ has the following probability distribution:

$$y_i^A[k]|\{r[l], s[l], m[l]\}_{l<k} = \begin{cases}
y_i^A[k-1], \text{ w.p. } \frac{n-1}{n}, \\
\dfrac{y_i^A[k-1](|\{l : r[l] = l, l \leq k\}|)}{|\{l : r[l] = l, l \leq k\}| + 1}, \\
\qquad \text{w.p. } \frac{\alpha}{n}\sum_{j\in\mathcal{N}_i} e_{ij} y_j^A[k-1] + \frac{1-\alpha}{n} h(b_i(A), b_i(B), \\
y_i^A[k-1]\dfrac{|\{l : r[l] = l, l \leq k\}|}{|\{l : r[l] = l, l \leq k\}| + 1}, \text{ otherwise.}
\end{cases}$$

Using this and the fact that probabilities are bounded by 1, we bound the expected value of $||n(k)||^2$ by:

$$\begin{aligned}
\mathbb{E}[||n[k]||^2|\{r[l], s[l], m[l]\}_{l<k}] &\leq ||Wy[k-1]||^2 + ||(I+W)y[k-1] + I||^2 + ||(I+W)y[k-1]||^2 \\
&\leq (3||W||^2 + 2||I||^2)||y[k-1]||^2 + ||I||^2.
\end{aligned}$$

Defining $K = \max(3||W||^2 + 2||I||^2, ||I||^2)$, we obtain

$$\mathbb{E}[||n[k]||^2|\{r[l], s[l], m[l]\}_{l<k}] \leq K(1 + y[k-1]) \leq \sup_{l<k} K(1 + ||y[l]||^2).$$

Finally, we can extend the above result for any sequence, which has a non-zero measure, *i.e.*,

$$\mathbb{E}[||n[k]||^2|\mathcal{F}[k-1] \leq K(1 + y[k-1]) \quad a.s.$$

■

Furthermore, since at each time period k, an agent i is chosen uniformly at random, it follows that $\lim_{k\to\infty}[D]_{ii}[k]/D_{jj}[k]=1$, for all $i,j \in \mathcal{N}$. Combining this with Lemma 3, we obtain that the awareness level vector $y^A[k]$ tracks a time-independent ordinary differential equation of the form (see Borkar (2008, Ch. 7)):

$$\frac{\nabla z^A(t)}{dt} = Wz^A(t) \quad \forall t>0, \tag{13}$$

with the initial condition $z^A(0)=y^A(0)$. The last three entries of vector $z^A(0)$ are 1, 0, and 0 respectively, and these elements are fixed for all $t>0$, since the last three rows of W in Equation (1) are zero vectors. Since the communication network is connected for all agents, the dynamical system described by Equation (13) converges to a well defined limit, and has a unique solution which does not depend on the initial conditions. If we denote the limit of the differential equation in Equation (13) by z^A_{lim}, then the awareness levels of agents converge a.s. to Borkar (2008, Ch. 7):

$$\lim_{k\to\infty} y^A[k] = z^A_{lim}. \tag{14}$$

Moreover, due to the specific structure of the differential equation, the awareness level can be characterized by:

$$z^A_{lim} = \lim_{t\to\infty} exp(W^t)z^A[0], \tag{15}$$

where $exp(.)$ is the matrix exponential of its argument. While Eq (15) uniquely characterizes the limit of agents' awareness levels, it is not particularly insightful. However, note that the dynamical system described by (13) defines a continuous time Markov Chain. We define the corresponding jump matrix V (Norris (1998, Ch. 2)) as:

$$[V]_{ij} = \begin{cases} \frac{[W]_{ij}}{-[W]_i} & \text{if } j\neq i,\ [W]_i \neq 0, \\ 0 & j\neq i,\ [W]_i = 0, \\ 0 & j=i,\ [W]_i \neq 0, \\ 1 & j=i,\ [W]_i = 0, \end{cases} \tag{16}$$

where $[W]_i = -\sum_{j\in\mathcal{N}, j\neq i}[W]_{ij}$. Using well-known results from the Markov Chain literature, we show that (Norris (1998, Ch. 3)):

$$\lim_{t\to\infty} exp(W^t) = \lim_{k\to\infty} V^k. \tag{17}$$

Combining Equations (14)-(17), we conclude that:

$$\lim_{k\to\infty} y^A[k] = \lim_{k\to\infty} V^k y^A[0].$$

Proof of Theorem 1

First we study the average awareness level of the population about brand A. Given the structure of V in Equation (2), we obtain:

$$\lim_{k\to\infty}(V)^k = \lim_{k\to\infty}\begin{bmatrix} F^k & \left(\sum_{l=0}^{k} F^l\right) E \\ 0 & I_3 \end{bmatrix} = \begin{bmatrix} 0 & (I-F)^{-1}E \\ 0 & I_3 \end{bmatrix}. \tag{18}$$

Moreover, using Proposition 1 and noting that $y^A_{n+1}[0]=1, y^A_{n+2}[0]=0, y^A_{n+3}[0]=0$, we obtain:

$$[y^A_{lim}]_{1:n} = (I-F)^{-1}[E]_1,$$

where $[E]_1$ is the first column of matrix E. Using the definition of the average awareness level and denoting by $\mathbf{1}$ the all ones vector, we get:

$$m^A_{lim} = \frac{1}{n}\mathbf{1}'(I-F)^{-1}[E]_1 = \frac{1}{n}\sum_{i\in\mathcal{N}}(1-\alpha)[\mathbf{1}'(I-F)^{-1}]_i \; h(b_i(A), b_i(B)).$$

The proof for the average awareness level about brand B follows from a similar argument and is therefore omitted.

Proof of Lemma 1

From our assumptions, function $h(x,y)$ is continuously differentiable in x for all $y\in[0,C]$. Moreover, all of the constraints in optimization problem (3) are affine in $\{b_i(A)\}_{i\in\mathcal{N}}$. Therefore, an optimal solution must satisfy the KKT conditions. Let $\{\lambda_i\}_{i\in\mathcal{N}}$ denote the KKT multipliers for the inequality constraints, $b_i(A)\geq 0$, for all $i\in\mathcal{N}$, and γ denote the KTT multiplier for the budget constraint, $\sum_{i\in\mathcal{N}} b_i(A)\leq C$. Then, the KKT conditions are given by:

$$\begin{aligned}
b^\star_i(A) &\geq 0 \text{ for all } i\in\mathcal{N}, \\
\sum_{i\in\mathcal{N}} b^\star_i(A) &\leq C, \\
\lambda_i &\geq 0 \text{ for all } i\in\mathcal{N}, \\
\lambda_i b^\star_i(A) &= 0 \text{ for all } i\in\mathcal{N}, \\
\gamma &\geq 0, \\
-\frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b^\star_i(A)} + \gamma - \lambda_i &= 0 \text{ for all } i\in\mathcal{N},
\end{aligned}$$

when $\{b^\star_i(A)\}_{i\in\mathcal{N}}$ is a solution to the optimization problem (3).

Proof of Proposition 2

The proof builds on the KKT conditions given in Lemma 1. Specifically, if $\gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}$ for a given i, then $b_i^\star(A) > 0$. Our claim follows from the fact that h function is assumed to be strictly concave in x and $\lambda_i \geq 0$. Moreover, if $b_i^\star(A) > 0$, then $\lambda_i = 0$ due to the complementary slackness condition. Therefore, $\gamma = \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)}$ for such i.

On the other hand, if $\gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}$, then $b_i^\star(A) = 0$, since otherwise $\lambda_i > 0$, which would violate the complementary slackness condition. Therefore, we have:

$$b_i^\star(A) = \begin{cases} 0 & \text{if } \gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \\ b_i(A) \text{ s.t. } \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{b=b_i(B)} = \gamma & \text{if } \gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \end{cases}$$

Moreover, $\sum_{i\in\mathcal{N}} b_i^\star(A) = C$ has to hold.

Proof of Proposition 3

Note that if $b_i^\star(A) = 0$, then $\gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(c,b_i(B))}{\partial x}\right|_{x=0}$ by Proposition 2. Since $c_i \geq c_j$, $b_i(B) = b_j(B)$, and $h(x,y)$ is increasing in x, then it must be that $\gamma \geq \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$. Therefore, $b_j^\star(A) = b_j^\star(A) = 0$.

On the other hand, if $b_i^\star(A) > 0$, then $\gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}$ by Proposition 2. For agent j, we either have $\gamma \geq \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$ or $\gamma < \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$. If the former condition holds, then $b_j^\star(A) = 0$, i.e., $b_i^\star(A) > b_j^\star(A)$. If the latter condition holds, then:

$$\frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)} = \gamma = \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=b_j^\star(A)},$$

by Proposition 2. The above equality implies that:

$$c_i\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)} = c_j\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=b_j^\star(A)}.$$

Finally, by noting that $h(x,y)$ is increasing and strictly concave in x for all $y \in [0,C]$, we obtain that $b_i^\star(A) \geq b_j^\star(B)$, which completes the proof of the proposition.

Proof of Proposition 4

We first note that strategy spaces $\mathcal{B}_A, \mathcal{B}_B$ are compact and convex subsets of $\mathbb{R}^N$. Moreover, $m_{lim}^A(b^A, b^B)$ and $m_{lim}^B(b^A, b^B)$ are continuous in b^A and b^B, respectively. Finally, since $h(x,y)$ is increasing in x for all $y \in [0,C]$, $h(x,y)$ is quasiconcave in x. Thus, a pure Nash equilibrium exists.

Proof of Proposition 5

The game has a symmetric equilibrium since it a symmetric game with compact, convex strategy spaces and continuous, quasiconcave utility functions (see Cheng et al. (2004)). In what follows, we show that the game under the assumptions in the proposition does not have asymmetric equilibria. For a given budget allocation of firm A, we define the activation set of A as the set of agents who receive non-zero budgets by A, i.e., $\mathcal{A}_A = \{i \in \mathcal{N} | b_i(A) > 0\}$. Similary, $\mathcal{A}_B$ denotes the activation set of firm B for a given budget allocation.

Assume for the sake of contradiction that in a given equilibrium, the activation set of firm A is a strict subset of the activation set of firm B, i.e., $\mathcal{A}_A \subset \mathcal{A}_B$. Then, there exists an $i \in \mathcal{N}$ such that $i \notin \mathcal{A}_A$ and $i \in \mathcal{A}_B$, and there exists an $j \in \mathcal{N}$ such that $j \in \mathcal{A}_A, \mathcal{A}_B$ and firm A allocates strictly larger budget to agent j than firm B (since both firms have equal budgets). If we denote by $b(A), b(B)$ the budget allocation vectors of firms A and B at this equilibrium, then $b_i(A) = 0$, $b_i(B), b_j(B), b_j(A) > 0$, and $b_j(A) > b_j(B)$.
From Lemma 1, the following must hold for i and j:

$$c_i \frac{\partial h(0, b_i(B))}{\partial x} \leq c_j \frac{\partial h(b_j(A), b_j(B))}{\partial x}, \tag{19}$$

$$c_i \frac{\partial h(b_i(B), 0)}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{20}$$

Using the fact that $\frac{\partial h(x,y)}{\partial x} > 0$, $c_i > 0 \ \forall \ i \in \mathcal{N}$, and dividing both sides of Eq. (19) by the corresponding sides of Eq. (20) we obtain

$$\frac{\partial h(0, b_i(B))}{\partial x} \frac{\partial h(b_j(B), b_j(A))}{\partial x} \leq \frac{\partial h(b_j(A), b_j(B))}{\partial x} \frac{\partial h(b_i(B), 0)}{\partial x}. \tag{21}$$

We first note that:

$$\frac{\partial h(b_i(B), 0)}{\partial x} < \frac{\partial h(0, 0)}{\partial x} \leq \frac{\partial h(0, b_i(B))}{\partial x}, \tag{22}$$

where the first inequality follows from the fact that $h(x, y)$ is strictly concave in x, and the second inequality follows from the fact that the mixed derivative is non-negative. Since $b_j(A) > b_j(B)$ and following the same argument as above, we show that:

$$\frac{\partial h(b_j(A), b_j(B))}{\partial x} < \frac{\partial h(b_j(B), b_j(B))}{\partial x} \leq \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{23}$$

From Eqs. (22) and (23):

$$\frac{\partial h(b_i(B), 0)}{\partial x} \frac{\partial h(b_j(A), b_j(B))}{\partial x} < \frac{\partial h(0, b_i(B))}{\partial x} \frac{\partial h(b_j(B), b_j(A))}{\partial x}.$$

However, the above equation contradicts with Eq. (21), and $b(A)$ and $b(B)$ cannot satisfy the KKT conditions. Therefore, $b(A), b(B)$ cannot be equilibrium budget allocations, and $\mathcal{A}_A \not\subset \mathcal{A}_B$. Using a similar argument, we can show that $\mathcal{A}_B \not\subset \mathcal{A}_A$.

Next for the sake of contradiction, we assume that in a given equilibrium, there exists an $i \in \mathcal{N}$ such that $i \notin \mathcal{A}_A$ and $i \in \mathcal{A}_B$, and there exists an $j \in \mathcal{N}$ such that $j \in \mathcal{A}_A$ and $j \notin \mathcal{A}_B$. If we denote $b(A), b(B)$ as the budget allocation vectors of firms A and B at this equilibrium, then $b_i(A) = b_j(B) = 0$, and $b_j(A), b_i(B) > 0$.

Due to the KKT conditions, the following holds for i and j:

$$c_i \frac{\partial h(0, b_i(B))}{\partial x} \leq c_j \frac{\partial h(b_j(A), 0)}{\partial x}, \tag{24}$$

$$c_j \frac{\partial h(0, b_j(A))}{\partial x} \leq c_i \frac{\partial h(b_i(B), 0)}{\partial x}. \tag{25}$$

Once again, note that:

$$\frac{\partial h(b_j(A), 0)}{\partial x} < \frac{\partial h(0, 0)}{\partial x} \leq \frac{\partial h(0, b_j(A))}{\partial x}, \tag{26}$$

where the first inequality follows from the fact that $h(x, y)$ is strictly concave in x, and the second inequality follows from the fact that the mixed derivative is non negative. Similarly, we can also show that:

$$\frac{\partial h(b_i(B), 0)}{\partial x} < \frac{\partial h(0, 0)}{\partial x} \leq \frac{\partial h(0, b_i(B))}{\partial x}. \tag{27}$$

From Eqs. (24) and (26):

$$c_j \frac{\partial h(b_j(A), 0)}{\partial x} < c_j \frac{\partial h(0, b_j(A))}{\partial x} \leq c_i \frac{\partial h(b_i(B), 0)}{\partial x}. \tag{28}$$

From Eqs. (25) and (27):

$$c_i \frac{\partial h(b_i(B), 0)}{\partial x} < c_i \frac{\partial h(0, b_i(B))}{\partial x} \leq c_j \frac{\partial h(b_j(A), 0)}{\partial x}. \tag{29}$$

However, Eqs. (28) and (29) contradict each other. Thus, $b(A)$ and $b(B)$ cannot satisfy the KKT conditions, and they cannot be part of an equilibrium budget allocation. Therefore, at any equilibrium $\mathcal{A}_A = \mathcal{A}_B$, i.e., the activation sets of A and B are equal.

Let's assume that given $\mathcal{A}_A = \mathcal{A}_B$, the budget allocations are asymmetric, i.e., there exists an $i \in \mathcal{N}$ for which $b_i(A) > b_i(B)$. Since the budgets for A and B are equal, there exists $j \in \mathcal{N}$ such that $b_j(B) > b_j(A)$. From Lemma 1, the following must hold for i and j:

$$c_i \frac{\partial h(b_i(A), b_i(B))}{\partial x} = c_j \frac{\partial h(b_j(A), b_j(B))}{\partial x}, \tag{30}$$

$$c_i \frac{\partial h(b_i(B), b_i(A))}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{31}$$

However,

$$\frac{\partial h(b_i(A), b_i(B))}{\partial x} < \frac{\partial h(b_i(B), b_i(B))}{\partial x} \leq \frac{\partial h(b_i(B), b_i(A))}{\partial x}, \tag{32}$$

where the first inequality follows from the fact that $h(x,y)$ is strictly concave in x, and the second inequality follows from the fact that the mixed derivative is non-negative. By noting the fact that $b_j(B) > b_j(A)$, and following the same argument as above, we can show that:

$$\frac{\partial h(b_j(B), b_j(A))}{\partial x} < \frac{\partial h(b_j(A), b_j(A))}{\partial x} \leq \frac{\partial h(b_j(B), b_j(B))}{\partial x}. \tag{33}$$

From Eqs. (30) and (32):

$$c_j \frac{\partial h(b_j(A), b_j(B))}{\partial x} = c_i \frac{\partial h(b_i(A), b_i(B))}{\partial x} < c_i \frac{\partial h(b_i(B), b_i(A))}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{34}$$

From Eqs. (31) and (33):

$$c_i \frac{\partial h(b_i(B), b_i(A))}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x} < c_j \frac{\partial h(b_j(A), b_j(A))}{\partial x} = c_i \frac{\partial h(b_i(A), b_i(B))}{\partial x}. \tag{35}$$

However, Eqs. (34) and (35) contradict each other. Thus, $b(A) = b(B)$, i.e., equilibria are symmetric under the assumptions of the proposition.

Proof of Proposition 6

As before, the game has a symmetric equilibrium since it a symmetric game with compact, convex strategy spaces and continuous, quasiconcave utility functions. It is straightforward to show that the symmetric equilibrium is unique. In particular, assume for the sake of contradiction that there exist two symmetric equilibria described by budget allocation vectors b^1 and b^2. Let i the smallest index for which $b_i^1 \neq b_i^2$ and assume that $b_i^1 > b_i^2$. Moreover, let $j > i$ such that $b_j^1 < b_j^2$. Note that

$$c_i \frac{\partial h(b_i^1, b_i^1)}{\partial x} \geq c_j \frac{\partial h(b_j^1, b_j^1)}{\partial x} \text{ and } c_i \frac{\partial h(b_i^2, b_i^2)}{\partial x} \leq c_j \frac{\partial h(b_j^2, b_j^2)}{\partial x}, \tag{36}$$

since b^1 and b^2 are equilibrium allocations. Moreover, from the assumptions of the proposition

$$\frac{\partial h(b_j^2, b_j^2)}{\partial x} < \frac{\partial h(b_j^1, b_j^1)}{\partial x} \text{ and } \frac{\partial h(b_i^1, b_i^1)}{\partial x} < \frac{\partial h(b_i^2, b_i^2)}{\partial x}, \tag{37}$$

since $b_i^1 > b_1^2$ and $b_j^1 < b_j^2$. However, Equations (36) and (37) lead to a contradiction. Finally, note that the assumptions in the proposition imply that

$$\frac{\partial h(t_1, t_2)}{\partial x} < \frac{\partial h(t_2, t_1)}{\partial x}, \text{ for } 0 \leq t_2 < t_1 \leq C.$$

Using this fact and similar arguments as those in the proof of Proposition 5 we can show that there exist no asymmetric equilibria.

Proof of Proposition 7

As before, it is straightforward to show that the game has a symmetric equilibrium. In the rest of the proof, we show that under the assumptions of Proposition 7 the game has asymmetric equilibria as well. Consider a *restricted* game in which we impose the following additional constraints: $b_1(B) = b_2(A) = 0$, where we assume without loss of generality that $c_1 \geq c_2 \geq \cdots \geq c_n$. Note that even after introducing these constraints, the strategy spaces of both players remain convex and compact subsets of $\mathbb{R}^N$, and their utility functions are continuous and quasiconcave. Therefore, the restricted game has an equilibrium. Denote by $\{b_i^\star(A)\}_{i\in\mathcal{N}}$, $\{b_i^\star(B)\}_{i\in\mathcal{N}}$ the budget allocations of firms A and B at an equilibrium of the restricted game. Then, vectors $\{b_i^\star(A)\}_{i\in\mathcal{N}}$ and $\{b_i^\star(B)\}_{i\in\mathcal{N}}$ have to satisfy the following:

$$c_2\ \frac{\partial h(b_2^\star(B),0)}{\partial x} = c_j \frac{\partial h(b_j^\star(B),b_j^\star(A))}{\partial x},\ \text{for all } j\neq 1,2 \text{ such that } b_j^\star(B)>0, \tag{38}$$

since firm B will definitely allocate a positive budget to individual 2. Our goal is to show that either the allocation $\{b_i^\star(A)\}_{i\in\mathcal{N}}$, $\{b_i^\star(B)\}_{i\in\mathcal{N}}$ is an equilibrium of the original game or we can construct an equilibrium of the original game using $\{b_i^\star(A)\}_{i\in\mathcal{N}}$, $\{b_i^\star(B)\}_{i\in\mathcal{N}}$. Assume for the sake of contradiction that this is not the case and firm B has a profitable deviation (we can rule out firm A having a profitable deviation using similar arguments). Firm B's profitable deviation has to involve agent 1, i.e., it should be that

$$c_1\ \frac{\partial h(0,b_1^\star(A))}{\partial x} > c_2 \frac{\partial h(b_2^\star(B),0)}{\partial x}. \tag{39}$$

We have to consider the following three cases:

(a) There exists an agent $j\neq 1,2$ such that $b_j^\star(A), b_j^\star(B) > 0$. Since the allocations $\{b_i^\star(A)\}_{i\in\mathcal{N}}$ and $\{b_i^\star(B)\}_{i\in\mathcal{N}}$ are an equilibrium for the restricted game we have:

$$c_1\ \frac{\partial h(b_1^\star(A),0)}{\partial x} = c_j \frac{\partial h(b_j^\star(A),b_j^\star(B))}{\partial x}, \tag{40}$$

and

$$c_2\ \frac{\partial h(b_2^\star(B),0)}{\partial x} = c_j \frac{\partial h(b_j^\star(B),b_j^\star(A))}{\partial x} < c_1\ \frac{\partial h(0,b_1^\star(A))}{\partial x}. \tag{41}$$

Equations (40) and (41) imply:

$$\frac{\partial h(0,b_1^\star(A))}{\partial x}\frac{\partial h(b_j^\star(A),b_j^\star(B))}{\partial x} > \frac{\partial h(b_1^\star(A),0)}{\partial x}\frac{\partial h(b_j^\star(B),b_j^\star(A))}{\partial x}. \tag{42}$$

Moreover, Equation (40) implies that $b_j^\star(A) + b_j^\star(B) < b_1^\star(A)$ under the assumptions of the proposition since $\partial h(b_j^\star(A), b_j^\star(B))/\partial x < \partial h(b_j^\star(A) + b_j^\star(B), 0)/\partial x$ and it has to be the case that $\partial h(b_j^\star(A), b_j^\star(B))/\partial x > \partial h(b_1^\star(A), 0)/\partial x$ otherwise firm A would have a profitable deviation in the restricted game. This observation together with Equation (42) leads to a contradiction since:

$$\frac{\partial h(b_j^\star(A),b_j^\star(B))/\partial x}{\partial h(b_j^\star(B),b_j^\star(A))/\partial x} < \frac{\partial h(b_j^\star(A)+b_j^\star(B),0)/\partial x}{\partial h(0,b_j^\star(A)+b_j^\star(B))/\partial x} < \frac{\partial h(b_1^\star(A),0)/\partial x}{\partial h(0,b_1^\star(A))/\partial x}.$$

(b) There exists an agent $j \neq 1,2$ such that $b_j^\star(A) > 0$ and $b_j^\star(B) = 0$. Note that since $c_1 \geq c_j$ it has to be the case that $b_1^\star(A) \geq b_j^\star(A)$. Moreover, since the allocations $\{b_i^\star(A)\}_{i\in\mathcal{N}}$ and $\{b_i^\star(B)\}_{i\in\mathcal{N}}$ are an equilibrium for the restricted game we have:

$$c_1 \, \frac{\partial h(b_1^\star(A),0)}{\partial x} = c_j \frac{\partial h(b_j^\star(A),0)}{\partial x} \tag{43}$$

and

$$c_j \frac{\partial h(0,b_j^\star(A))}{\partial x} \leq c_2 \, \frac{\partial h(b_1^\star(B),0)}{\partial x} < c_1 \, \frac{\partial h(0,b_1^\star(A))}{\partial x}. \tag{44}$$

However, Equations (43) and (44) together with $b_j^\star(A) \leq b_1^\star(A)$ lead to a contradiction.

(c) Finally, we have to consider the case when $b_1^\star(A) = C$ and $b_j^\star(A) = 0$ for all $j \neq 1$. Then, we claim that there exists an equilibrium in the original (unrestricted) game with $b_1(A) = C$ and $b_1(B) < C$. First, note that $b_1(A) = C = b_1(B) = C$ cannot be an equilibrium since the activation sets for firms A, B in the symmetric equilibrium of the original game have cardinality at least two. Consider firm B's best response, which we denote by b^{BR}, to the allocation b^A such that $b_1(A) = C$ and $b_2(A) = \cdots = b_n(A) = 0$ from firm A. Then, we claim that the allocations b^A and b^{BR} are an equilibrium. Assume that this is not the case. Then, firm A has to have a profitable deviation. Note that it has to be the case that $0 < b_1^{BR}(B) < C$, since otherwise firm B would not have a profitable deviation from profile $b^\star$ and the equilibrium of the restricted game would be an equilibrium for the original game. But then $\frac{\partial h(b_1^{BR}(B),C)}{\partial x} < \frac{\partial h(C,b_1^{BR}(B))}{\partial x}$ and thus firm A has no profitable deviation (since otherwise firm B would have a profitable deviation as well).

Proof of Theorem 2

From Proposition 2, we have that $b_i^\star(A)$ is given by

$$\frac{1}{n} c_i(1-\alpha) \frac{\partial h(x,b_i(B))}{\partial x}\Big|_{x=b_i^\star(A)} = \gamma, \tag{45}$$

for an agent i that receives a positive advertising budget. For $h(x,y) = \frac{x}{x+y+\beta}$, Equation (45) takes the following form:

$$c_i \frac{b_i(B)+\beta}{(b_i^\star(A)+b_i(B)+\beta)^2} = \gamma',$$

where $\gamma' = \frac{n\gamma}{1-\alpha}$. After straightforward algebra we obtain

$$b_i^\star(A) = \frac{\sqrt{c_i}\sqrt{b_i(B)+\beta}}{\sqrt{\gamma'}} - b_i(B) - \beta.$$

Moreover, using $\sum_{i\in\mathcal{N}} b_i^\star(A) = C$ and $b_i(B) = C/n$, we get

$$\frac{\sqrt{C/n+\beta}}{\sqrt{\gamma'}} = \frac{2C+n\beta}{\sum_{i\in\mathcal{N}}\sqrt{c_i}},$$

and, thus,

$$b_i^\star(A) = \frac{\sqrt{c_i}\,(2C+n\beta)}{\sum_{i\in\mathcal{N}}\sqrt{c_i}} - \frac{C}{n} - \beta. \tag{46}$$

Condition (7) implies equation (45) is satisfied for all agents. By replacing the expression for $b_i^\star(A)$ from (46) to the objective function for the optimization problem of firm A, i.e., the problem described in (3), we obtain that

$$m^A_{lim} = \frac{1-\alpha}{n}\sum_{i\in\mathcal{N}} c_i - \left(\sum_{i\in\mathcal{N}}\sqrt{c_i}\right)\frac{1-\alpha}{n}\sum_{i\in\mathcal{N}}\sqrt{c_i}\frac{C/n+\beta}{2C+n\beta}.$$

Finally, by noting that $\sum_{i\in\mathcal{N}} c_i$ is fixed for any connected network and equal to $\frac{n}{1-\alpha}$, and by letting $b_i(B)=\frac{C}{n}$ we obtain

$$m^A_{lim} = 1 - \frac{1-\alpha}{n^2}\frac{C+n\beta}{2C+n\beta}\left(\sum_{i\in\mathcal{N}}\sqrt{c_i}\right)^2. \tag{47}$$

Proof of Theorem 3

To show a lower bound for Π, we provide an adjacency matrix and a corresponding equilibrium that yields a ratio of awareness levels equal to $\frac{h'(C,0)}{h'(0,C)}$. In particular, a population of n agents are organized in a social network structure characterized by the following matrix $\mathcal{E}$:

$$\begin{aligned}
\mathcal{E}_{11} = \mathcal{E}_{12} = 0, \mathcal{E}_{13} = \cdots = \mathcal{E}_{1n} &= \frac{1}{n-2}\\
\mathcal{E}_{21} = \mathcal{E}_{22} = 0, \mathcal{E}_{13} = \cdots = \mathcal{E}_{1n} &= \frac{1}{n-2}\\
\mathcal{E}_{31} = \cdots = \mathcal{E}_{n1} = \frac{\lambda}{1+\lambda}, \mathcal{E}_{32} = \cdots = \mathcal{E}_{n2} &= \frac{1}{1+\lambda}\\
\mathcal{E}_{ij} &= 0, \text{ for } i,j>2,
\end{aligned}$$

where $\lambda > 1$ is a constant. In particular, agents 1 and 2 are weighted disproportionately more compared to agents $3,\cdots,n$ in adjacency matrix $\mathcal{E}$ and agent 1 is weighted λ times as much as agent 2. From the definition of the centrality vector ($\mathbf{c} = [\mathbf{1}'(I-F)^{-1}]$) and straightforward algebra we obtain that:

$$\frac{c_1}{c_2} = \frac{\frac{\alpha\lambda}{1+\lambda} + \frac{1}{c_3(n-2)}}{\frac{\alpha}{1+\lambda} + \frac{1}{c_3(n-2)}}$$

and

$$\frac{c_3}{c_2} = \frac{\alpha}{n-2}\frac{c_1}{c_2} + \frac{\alpha}{n-2} + \frac{1}{c_2}.$$

We choose λ so that $\frac{c_1}{c_2} = \frac{h'(C,0)}{h'(0,C)}$ (this is always possible) and consider the following allocation: $b_1(A) = C = b_2(B)$ and $b_2(A) = \cdots = b_n(A) = b_1(B) = b_3(B) = \cdots = b_n(B) = 0$. Then, it is straightforward to see that neither of the two firms has any profitable deviation that involves moving advertising funds between agents 1 and 2. This is due to the fact that $c_1 = \frac{h'(C,0)}{h'(0,C)}c_2$. Thus, for the allocation to be an equilibrium we need to make sure that firm B (and as a consequence firm A) has no incentive to deviate by moving advertising funds to agents $3, \cdots, n$. Recall that $h'(0,0) < \epsilon$, where $\epsilon > 0$ is a constant. Thus, for the allocation to be an equilibrium, the following condition is sufficient: $c_2 h'(C,0) \geq \epsilon \cdot c_3$. Note that for a sufficiently large population of agents, the condition is met (since as we increase the number of agents n, the ratio c_3/c_2 decreases). Thus, the proposed allocation is an equilibrium and $\Pi \geq \frac{c_1 h(C,0)}{c_2 h(C,0)} = \frac{h'(C,0)}{h'(0,C)}$.

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References

Acemoglu, D, K Bimpikis, A Ozdaglar. 2012. Dynamics of information exchange in endogenous social networks. *Forthcoming in Theoretical Economics* .

Ballester, C, A Calvo-Armengol, Y Zenou. 2006. Who's who in networks. wanted: the key player. *Econometrica* **74**(5) 1403–1417.

Bass, Frank M., Norris Bruce, Sumit Majumdar, B. P. S. Murthi. 2007. Wearout effects of different advertising themes: A dynamic bayesian model of the advertising-sales relationship. *Marketing Science* **26**(2) 179–195.

Bergemann, D, J Valimaki. 1996. Learning and strategic pricing. *Econometrica* **64**(5) 1125–1150.

Bergemann, D, J Valimaki. 2000. Experimentation in markets. *Review of Economic Studies* **67**(2) 213–234.

Bonacich, Phillip. 1987. Power and centrality: A family of measures. *American journal of sociology* 1170–1182.

Borkar, V S. 2008. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press.

Bulow, Jeremy I, John D Geanakoplos, Paul D Klemperer. 1985. Multimarket oligopoly: Strategic substitutes and complements. *The Journal of Political Economy* **93**(3) 488–511.

Candogan, O, K Bimpikis, A Ozdaglar. 2012. Optimal pricing in networks with externalities. *Operations Research* **60**(4) 883–905.

Cheng, Shih-Fen, Daniel M Reeves, Yevgeniy Vorobeychik, Michael P Wellman. 2004. Notes on equilibria in symmetric games. *In Proceedings of Workshop on Game Theory and Decision Theory*.

Clifford, P, A Sudbury. 1973. A model for spatial conflict. *Biometrika* **60**(4) 581–588.

Conley, T G, C R Udry. 2010. Learning about a new technology: Pineapple in ghana. *American Economic Review* **100**(1) 35–69. doi:10.1257/aer.100.1.35.

DeGroot, M.H. 1974. Reaching a consensus. *Journal of American Statistical Association* **69** 118–121.

Dodson, Joe A, Eitan Muller. 1978. Models of new product diffusion through advertising and word-of-mouth. *Management Science* **24**(15) 1568–1578.

Fazeli, Arastoo, Ali Jadbabaie. 2012. Targeted marketing and seeding products with positive externality. *50th Annual Allerton Conference on Communication, Control, and Computing*. 1111–1117.

Godes, D, D Mayzlin. 2004. Using online conversations to study word-of-mouth communication. *Marketing Science* **23**(4) pp. 545–560.

Goyal, S, M Kearns. 2012. Competitive contagion in networks. *Proceedings of the 44th ACM Symposium on Theory of Computing (STOC)* .

Hartline, J, V Mirrokni, M Sundararajan. 2008. Optimal marketing strategies over social networks. *Proceedings of the 17th international conference on World Wide Web* .

Holley, R A, T M Liggett. 1975. Ergodic theorems for weakly interacting infinite systems and the voter model. *The Annals of Probability* **3**(4) 643–663.

Ifrach, Bar, Costis Maglaras, Marco Scarsini. 2013. Monopoly pricing in the presence of social learning. *Working paper* .

Iyer, G, D Soberman, J M Villas-Boas. 2005. The targeting of advertising. *Marketing Science* **24**(3) 461–476.

Jackson, Matthew O. 2008. *Social and economic networks*. Princeton University Press.

Kempe, D, J Kleinberg, É Tardos. 2003. Maximizing the spread of influence through a social network. *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* .

Laurent, Gilles, Jean-Noel Kapferer, Francoise Roussel. 1995. The underlying structure of brand awareness scores. *Marketing Science* **14**(3) 170–179.

Mahajan, Vijay, Eitan Muller, Subhash Sharma. 1984. An empirical comparison of awareness forecasting models of new product introduction. *Marketing Science* **3**(3) 179–197.

Naik, Prasad A, Murali K Mantrala, Alan G Sawyer. 1998. Planning media schedules in the presence of dynamic advertising quality. *Marketing Science* **17**(3) 214–235.

Naik, Prasad A, Ashutosh Prasad, Suresh P Sethi. 2008. Building brand awareness in dynamic oligopoly markets. *Management Science* **54**(1) 129–138.

Norris, J R. 1998. *Markov Chains (Cambridge Series in Statistical and Probabilistic Mathematics)*. Cambridge University Press.

Pauwels, Koen. 2004. How dynamic consumer response, competitor response, company support, and company inertia shape long-term marketing effectiveness. *Marketing Science* **23**(4) 596–610.

Tullock, G. 1980. Efficient Rent Seeking. J. M. Buchanan, R. D. Tollison, G. Tullock, eds., *Toward a Theory of the Rent Seeking Society*. Texas A&M University Press, 97–112.

Yildiz, E, D Acemoglu, A Ozdaglar, A Saberi, A Scaglione. 2012. Discrete opinion dynamics with stubborn agents. *Working paper* .