A Projection Framework for Near-Potential Games

Ozan Candogan, Asuman Ozdaglar and Pablo A. Parrilo

***Abstract*— Potential games are a special class of games that admit tractable static and dynamic analysis. Intuitively, games that are "close" to a potential game should enjoy somewhat similar properties. This paper formalizes and develops this idea, by introducing a systematic framework for finding potential games that are close to a given arbitrary strategic-form finite game. We show that the sets of exact and weighted potential games (with fixed weights) are subspaces of the space of games, and that for a given game, the closest potential game in these subspaces (possibly subject to additional constraints) can be found using convex optimization. We provide closed-form solutions for the closest potential game in these subspaces, and extend our framework to more general classes of games.**

We further investigate and quantify to what extent the static and dynamic features of potential games extend to "near-potential" games. In particular, we show that for a given strategic-form game, we can characterize its approximate equilibria and the sets to which better-response dynamics converges to, as a function of the distance of the game to its potential approximation.

I. Introduction

Potential games are a class of games with appealing static and dynamic properties. For instance, in such games pure-strategy Nash equilibria always exist, and many of the simple user dynamics (e.g., fictitious play) converge to a Nash equilibrium [1]–[3]. Because of these properties, potential games found numerous applications in various control and resource allocation problems, e.g., [1], [4]–[6]. However, many multi-agent strategic interactions in engineering and economics literatures cannot be directly modelled as a potential game.

Intuitively, games that are "close" to potential games, should inherit some of these static and dynamic properties. In this work, our goal is to provide a systematic framework for identifying a close potential game to some given game, and to study to which extent the properties of this potential game extend to the original game. For this purpose, we focus, in increased order of generality, on three well-known classes of potential games: exact potential games, weighted potential games, and ordinal potential games.

We first characterize the geometry of the problem, by formally defining the vector space of all games, and introducing a natural inner product structure on it. We show that the set of exact potential games, and the set of weighted potential games with fixed weights are *subspaces* of the space of games (Theorem 2). Therefore, for any given game, the closest exact (or fixed-weight) potential game can be obtained by projection onto the relevant subspace. On the other hand, the set of all weighted potential games, and similarly the set of ordinal potential games, are nonconvex sets. Hence, finding the closest weighted and ordinal potential games to a given game requires solving nonconvex optimization problems.

For any finite game, we provide a *closed-form solution* for the closest fixed-weight potential game, by projecting the original game to the subspace of weighted potential games associated with these weights (Theorem 3). If the weights are unknown, the underlying set is no longer convex. Nevertheless, we present a related convex optimization formulation for finding close weighted potential games (Section IV-B). Although the game obtained through this approach will not necessarily be the closest weighted potential game to the original game, examples show that it often is a very good approximation and yields a weighted potential game whose distance (in terms of utility differences) to the original game is much smaller than that of the closest exact potential game.

Additionally, we show that the approximate equilibria and the better-response dynamics in arbitrary strategic-form finite games can be analyzed using the close weighted and ordinal potential games suggested by our framework (cf. Propositions 6 and 7). The main idea behind this approach is to use the "distance" of the original game from the set of potential games to approximately establish the properties of the original game. Our results indicate that the equilibria of the close potential game can be used to characterize the approximate equilibria of the original game, and the sets to which the update rules converge. Moreover, the closer the original game is to a potential game, the tighter our characterization becomes.

The remainder of this paper is organized as follows: In Section II we present relevant game-theoretic background. In Section III, we establish the geometric properties of the sets of potential games. We discuss different formulations for finding close weighted and ordinal potential games to a given game in Section IV, followed by a numerical example in Section V. In Section VI, we show how this framework can be used to establish static and dynamic properties of a given near-potential game. We close in Section VII with concluding remarks and future directions.

II. Preliminaries

In this section we present the game theoretic background that is relevant to our work.

Throughout this paper we consider strategic-form finite games. A (noncooperative) finite game in strategic form

This research was funded in part by National Science Foundation grants DMI-0545910, ECCS-0621922, by the DARPA ITMANET program, and by the AFOSR MURI R6756-G2.

All authors are with the Laboratory of Information and Decision Systems, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Room 32-D608 Cambridge, MA 02139.

emails: {candogan,asuman,parrilo}@mit.edu

consists of:

- A finite set of players, denoted by $\mathcal{M} = \{1, \dots M\}$.
- Strategy spaces: A finite set of strategies (or actions) E^m, for every $m \in \mathcal{M}$.
- Utility functions: $u^m : E \to \mathbb{R}$, for every $m \in \mathcal{M}$.

A (strategic-form) game instance is accordingly given by the tuple $\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$.

We assume that each player in a game has at least two strategies: $|E^m| \geq 2$, for all $m \in \mathcal{M}$. The joint strategy space of a game is denoted by $E = \prod_{m\in\mathcal{M}} E^m$. We refer to a collection of strategies of all players as a *strategy profile* and denote it by $\mathbf{p} = (p^1, \dots, p^M) \in E$. The strategies of all players but the mth one is denoted by $\mathbf{p}^{-m}$.

The basic solution concept in a noncooperative game is that of a Nash Equilibrium (NE). A (pure) Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally, a strategy profile $\mathbf{p} = (p^1, \dots, p^M)$ is a Nash equilibrium if

$$u^m(p^m, \mathbf{p}^{-m}) \geq u^m(q^m, \mathbf{p}^{-m}),$$

for every $q^m \in E^m$ and $m \in \mathcal{M}$.

To address strategy profiles that are approximately a Nash equilibrium, we use the concept of ϵ-equilibrium. A strategy profile $\mathbf{p} = (p^1, \dots, p^M)$ is an ϵ-equilibrium if

$$u^m(p^m, \mathbf{p}^{-m}) \geq u^m(q^m, \mathbf{p}^{-m}) - \epsilon$$

for every $q^m \in E^m$ and $m \in \mathcal{M}$. Note that a Nash equilibrium is an ϵ-equilibrium with $\epsilon = 0$.

We next define potential games [1], which are central to our discussion in the subsequent sections.

Definition 2.1 (Potential Games): Consider a noncooperative game $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$. If there exists a function $\phi : E \to \mathbb{R}$ such that for every $m \in \mathcal{M}$, $p^m, q^m \in E^m$, $\mathbf{p}^{-m} \in E^{-m}$,

1) $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m})$, then $\mathcal{G}$ is an *exact potential game*.
2) $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = w_m(u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}))$, for some strictly positive weight $w_m > 0$, then $\mathcal{G}$ is a *weighted potential game*.
3) $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) > 0 \Leftrightarrow u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) > 0$, then $\mathcal{G}$ is an *ordinal potential game*.

The function ϕ is referred to as a *potential function* of the game. This definition suggests that potential games are games in which the interests of players are captured by a global potential function ϕ.

Note that every exact potential game is a weighted potential game with $w_m = 1$ for all $m \in \mathcal{M}$. From the definitions it also follows that every weighted potential game is an ordinal potential game. In other words, ordinal potential games generalize weighted potential games, and weighted potential games generalize exact potential games.

We denote the sets of exact, weighted, and ordinal potential games by $\mathcal{P}$, $\mathcal{WP}$, $\mathcal{OP}$ respectively. In order to characterize $\mathcal{WP}$, it is sufficient to consider weights $w_m \geq 1$. This follows since in any weighted potential game, the potential function and the weights can be jointly scaled by a positive scalar to obtain a different potential function and larger weights. Given a fixed set of weights $w = \{w_m\}$, we refer to the set of all weighted potential games with these weights as *fixed-weight* potential games, and denote this set by $\mathcal{P}_w$. In particular, the set of exact potential games is equivalent to $\mathcal{P}_1$. It can be seen that the set of weighted potential games can be expressed as a union of the sets of fixed-weight potential games over weights $w_m \geq 1$ for all $m \in \mathcal{M}$, i.e. $\mathcal{WP} = \cup_{w\geq 1} \mathcal{P}_w$.

We conclude this section by providing necessary and sufficient conditions for a game to be an exact or ordinal potential game. We first introduce some key concepts used in establishing these conditions.

Definition 2.2 (Path – Closed Path – Improvement Path): A path is a collection of strategy profiles $\gamma = (\mathbf{p}_0, \dots \mathbf{p}_N)$ such that $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ differ in the strategy of exactly one player. A path is a closed path (or a cycle) if $\mathbf{p}_0 = \mathbf{p}_N$. A path is an improvement path if $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$ where m_i is the player who modifies its strategy when the strategy profile is updated from $\mathbf{p}_{i-1}$ to $\mathbf{p}_i$.

The transition from strategy profile $\mathbf{p}_{i-1}$ to $\mathbf{p}_i$ is referred to as step i of the path. We refer to a closed improvement path such that the inequality $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$ is strict for at least a single step of the path, as a *weak improvement cycle*. We say that a closed path is *simple* if no strategy profile other than the first and the last strategy profiles is repeated along the path. For any path $\gamma = (\mathbf{p}_0, \dots \mathbf{p}_N)$, let $I(\gamma)$ represent the "utility improvement" along the path. Namely

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(\mathbf{p}_i) - u^{m_i}(\mathbf{p}_{i-1}),$$

where m_i is the index of the player that modifies its strategy in the ith step of the path.

The following proposition provides an alternative characterization of exact and ordinal potential games. This characterization will be used in studying the geometry of sets of different classes of potential games (cf. Theorem 2).

Proposition 1 ([1], [7]): (i) A finite game $\mathcal{G}$ is an exact potential game if and only if $I(\gamma) = 0$ for all simple closed paths γ.

(ii) A finite game $\mathcal{G}$ is an ordinal potential game if and only if it does not include weak improvement cycles.

III. Sets of Potential Games

In this section, we investigate the properties and the geometry of potential games. In particular, we show that the sets of weighted and ordinal potential games are nonconvex subsets of the space of games, but the set of exact potential games is a subspace.

We first provide a formal definition of the space of games. Consider games with set of players $\mathcal{M}$, set of strategy profiles $E \stackrel{\triangle}{=} E^1 \times \dots \times E^M$. We denote by $C_0 = \{f | f : E \to \mathbb{R}\}$, the set of utility functions that can be present in a game with set of strategy profiles E. Since every function $f \in C_0$ has a finite domain, it can be represented as a vector in $\mathbb{R}^{|E|}$. Note that every utility function in a game with set of strategy profiles E and set of players $\mathcal{M}$, belongs to C_0, i.e., $u^m \in C_0$ for all $m \in \mathcal{M}$. We denote the set of all games with set of players $\mathcal{M}$ and set of strategy profiles E by $\mathcal{G}_{\mathcal{M},E}$. Any two games in this space differ only in their payoff functions, thus this space can alternatively be identified by $C_0^{|\mathcal{M}|} \approx \mathbb{R}^{|E||\mathcal{M}|}$, i.e., the set of all payoff functions that define such games.

Before we present our result on the sets of potential games, we introduce the notion of convexity for sets of games.

Definition 3.1: Let $B \subset \mathcal{G}_{\mathcal{M},E}$. The set B is said to be convex if and only if for any two game instances $\mathcal{G}_1, \mathcal{G}_2 \in B$ with collections of utilities $u = \{u^m\}_{m\in\mathcal{M}}$, $v = \{v^m\}_{m\in\mathcal{M}}$ respectively

$$\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{\alpha u^m + (1-\alpha)v^m\}_{m\in\mathcal{M}}\rangle \in B,$$

for all $\alpha \in [0, 1]$.

Below we show that the set of exact potential games is a subspace of the space of games, whereas, the sets of weighted and ordinal potential games are nonconvex.

Theorem 2: (i) The sets of exact potential games, $\mathcal{P}$, and fixed-weight potential games, $\mathcal{P}_w$, are subspaces of $\mathcal{G}_{\mathcal{M},E}$.

(ii) The sets of weighted potential games, $\mathcal{WP}$, and ordinal potential games, $\mathcal{OP}$, are nonconvex subsets of $\mathcal{G}_{\mathcal{M},E}$.

Proof: (i) Proposition 1 (i) implies that the set of exact potential games is the subset of space of games where the utility functions satisfy the condition $I(\gamma) = 0$ for every simple closed path γ. Note that for each γ, $I(\gamma) = 0$ is a linear equality constraint on the utility functions $\{u^m\}$. Thus, the set of exact potential games is the intersection of the sets defined by these linear equality constraints. Therefore, it is a subspace of the space of games, $\mathcal{G}_{\mathcal{M},E} \approx C_0^{|\mathcal{M}|}$.

It can be seen from Definition 2.1 that similar to exact potential games, fixed-weight potential games are characterized by linear equality constraints. Thus, the proof for $\mathcal{P}_w$ follows similarly.

(ii) We prove the claim by showing that the convex combination of two weighted potential games is not an ordinal potential game. This implies that the sets of both weighted and ordinal potential games are nonconvex since every weighted potential game is an ordinal potential game.

In Table I we present the payoffs and the potential in a two player game, $\mathcal{G}_1$, where each player has two strategies. Given strategies of both players the first table shows payoffs of players (the first number denotes the payoff of the first player), the second table shows the corresponding potential function. In both tables the first column stands for actions of the first player and the top row stands for actions of the second player. Note that this game is a weighted potential game with weights $w_1 = 1$, $w_2 = 3$.

(u^1, u^2)	A	B
A	0,0	0,4
B	2,0	8,6

ϕ	A	B
A	0	12
B	2	20

TABLE I

PAYOFFS AND POTENTIAL IN $\mathcal{G}_1$

Similarly, another game $\mathcal{G}_2$ is defined in Table II. Note that this game is also a weighted potential game with weights $w_1 = 3$, $w_2 = 1$.

(u^1, u^2)	A	B
A	4,2	6,0
B	0,8	0,0

ϕ	A	B
A	20	18
B	8	0

TABLE II

PAYOFFS AND POTENTIAL IN $\mathcal{G}_2$

In Table III, we consider a game $\mathcal{G}_3$, in which the payoffs are averages (hence convex combinations) of payoffs of $\mathcal{G}_1$ and $\mathcal{G}_2$.

(u^1, u^2)	A	B
A	2,1	3,2
B	1,4	4,3

TABLE III

PAYOFFS IN $\mathcal{G}_3$

Note that this game has a weak improvement cycle:

$$(A, A) - (A, B) - (B, B) - (B, A) - (A, A).$$

From Proposition 1 (ii), it follows that $\mathcal{G}_3$ is not an ordinal potential game.

The above example shows that the sets of weighted and ordinal potential games with two players each of which has two strategies is nonconvex. For general n player games, the claim immediately follows by constructing two n player weighted potential games, and embedding $\mathcal{G}_1$ and $\mathcal{G}_2$ in these games. The details are omitted due to page constraints. ■

We next discuss the geometry of the sets of weighted and ordinal potential games. The above theorem together with $\mathcal{WP} = \cup_{w\geq 1}\mathcal{P}_w$, implies that the set of weighted potential games is an uncountable union of subspaces of $\mathcal{G}_{\mathcal{M},E}$. Given an ordinal potential game, the multiplication of the utilities by a positive scalar gives a game with the same weak improvement cycles. Since the original game is an ordinal potential game and does not have a weak improvement cycle, the scaled game cannot have a weak improvement cycle. Thus, the scaled game is an ordinal potential game as well. Therefore, we conclude that $\mathcal{OP}$ is a cone. However, Theorem 2 implies that this is not a convex cone.

IV. PROJECTION TO THE SET OF POTENTIAL GAMES

In this section, we develop a framework for finding close weighted and ordinal potential games to a given game. It was shown in Section III that the fixed-weight potential games and exact potential games form subspaces. In Section IV-A, we develop a projection framework which gives closed form solution for the closest fixed-weight potential game to a given game. In Section IV-B, we discuss methods for choosing weights to obtain better weighted potential game approximations. In Section IV-C, we establish similar results for ordinal potential games.

A. Fixed-Weight Potential Games

In this section, we obtain closed form solution of the closest fixed-weight potential game to a given game. Before presenting our results, we first introduce some necessary operators and notation.

For each player $m \in \mathcal{M}$, we define the difference operator D_m such that for all $f : E \to \mathbb{R}$ and $\mathbf{p}, \mathbf{q} \in E$ that differ in the strategy of only player m,

$$(D_m f)(\mathbf{p}, \mathbf{q}) = f(\mathbf{q}) - f(\mathbf{p}),$$

and $(D_m f)(\mathbf{p}, \mathbf{q}) = 0$ otherwise. The difference operators allow for alternative definitions of potential games: A game is a weighted potential game with weights $\{w_m\}$, if there exists a potential function ϕ, such that $D_m u^m = w_m D_m \phi$ for all $m \in \mathcal{M}$ (it is an exact potential game if all weights are equal to 1).

As mentioned earlier, the set of functions with domain E, is denoted by C_0 and this set can equivalently be represented as a vector in $\mathbb{R}^{|E|}$. Motivated by this, we use the regular inner product of $R^{|E|}$ in C_0, i.e., the inner product of any $f_1, f_2 \in C_0$ is given by $\langle f_1, f_2 \rangle = \sum_{\mathbf{p} \in E} f_1(\mathbf{p}) f_2(\mathbf{p})$.

For each player m, we define the set $A_m = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p}, \mathbf{q} \in E$ differ in the strategy of only player $m\}$ and we denote the union of these sets by $A = \cup_m A_m$. We refer to functions $X : E \times E \to \mathbb{R}$ such that $X(\mathbf{p}, \mathbf{q}) = X(\mathbf{q}, \mathbf{p}) \in \mathbb{R}$ if $(\mathbf{p}, \mathbf{q}) \in A$ and $X(\mathbf{p}, \mathbf{q}) = 0$ otherwise as the pairwise ranking functions. We denote the space of all such functions by C_1 and associate with it the inner product $\langle X, Y \rangle = \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q})$, where $X, Y \in C_1$. For all $m \in \mathcal{M}$ and $f \in C_0$, it can be seen that $(D_m f) \in C_1$.

The adjoint of D_m is the unique operator D_m^* which satisfies,

$$\langle X, D_m f \rangle = \langle D_m^* X, f \rangle,$$

for all $f : E \to \mathbb{R}$, $X : E \times E \to \mathbb{R}$. Using the definitions of the inner products and the difference operators, it follows that for all $X \in C_1$ the operator D_m^* satisfies,

$$(D_m^* X)(\mathbf{p}) = -\sum_{\mathbf{q} | (\mathbf{p}, \mathbf{q}) \in A_m} X(\mathbf{p}, \mathbf{q}).$$

Three operators that are related to the difference operator and its adjoint are $\Delta_{0,m} = D_m^* D_m$, $\Delta_0 = \sum_m \Delta_{0,m}$ and $\Pi_m = D_m^\dagger D_m$, where $\dagger$ denotes the pseudo inverse. The first two operators correspond to Laplacian operators on a graph associated with the game and the third operator is a projection operator to the orthogonal complement of the kernel of the difference operator D_m (for details see [8]). Some identities these operators satisfy, are presented in the next lemma. The proof is omitted due to space constraints and can be found in [8].

Lemma 1: (i) $\Delta_{0,m} = h_m \Pi_m$, (ii) $D_m \Pi_m = D_m$, (iii) $\ker \Delta_0 = \{f \in C_0 | f(\mathbf{p}) = c$ for all $\mathbf{p} \in E$ and some $c \in \mathbb{R}\}$.

We next present an optimization problem that finds the closest weighted potential game with weights $\{w_m\}$ to a given game. We quantify the distance between two games with utility functions $\{u^m\}$ and $\{\hat{u}^m\}$ as $\sum_{m \in \mathcal{M}} h_m ||u^m - \hat{u}^m||^2$, where we use the norm induced by the inner product in the space of payoff functions C_0, and $h_m = |E^m|$. It can be seen that $\sum_{m \in \mathcal{M}} h_m ||u^m||^2$ corresponds to a weighted l_2 norm in the space of games $\mathcal{G}_{\mathcal{M},E}$ (see [8]). Given a game with payoff functions $\{u^m\}$, the closest weighted potential game with weights $\{w_m\}$ and utility functions $\{\hat{u}^m\}$ is the solution of the following least-squares optimization problem:

$$\text{(P1 :)} \quad \begin{aligned} \min_{\phi, \{\hat{u}^m\}} \quad & \sum_m h_m ||u^m - \hat{u}^m||^2 \\ s.t. \quad & D_m \phi = w_m D_m \hat{u}^m, \\ & \text{for all } m \in \mathcal{M}. \end{aligned}$$

Since all the constraints are linear, the feasible set of $\{\hat{u}^m\}$ in this optimization problem is a subspace of $C_0^{|\mathcal{M}|}$. Thus, the optimal $\{\hat{u}^m\}$ is unique and given by the projection of the original game to the feasible subspace with respect to the norm in the objective function. The next theorem characterizes the optimal solution of this problem.

Theorem 3: The optimal solution of P1 is given by $\phi = \left(\sum_m \frac{1}{w_m^2} \Delta_{0,m}\right)^\dagger \sum_m \frac{1}{w_m} \Delta_{0,m} u^m$ and $\hat{u}^m = \frac{1}{w_m} \Pi_m \phi + (I - \Pi_m) u^m$ for all players $m \in \mathcal{M}$.

Proof: Since D_m is a linear operator, for any feasible $\hat{u}^m$ there exists two orthogonal components a^m, b^m such that a^m belongs to the kernel of D_m, b_m belongs to the orthogonal complement of the kernel of D_m and $\hat{u}^m = a^m + b^m$. Observe that Π_m is a projection operator to the orthogonal complement of the kernel of D_m and $I - \Pi_m$ is the projection operator to the kernel of D_m, where I stands for the identity operator. Using this fact, and the definitions of a^m and b^m, it can be seen that

$$\begin{aligned} ||u^m - \hat{u}^m||^2 & \\ &= ||\Pi_m(u^m - \hat{u}^m)||^2 + ||(I - \Pi_m)(u^m - \hat{u}^m)||^2 \\ &= ||\Pi_m u^m - b^m||^2 + ||(I - \Pi_m) u^m - a^m||^2. \end{aligned}$$

Therefore, P1 can be rewritten as:

$$\begin{aligned} \min_{\phi, \{a^m\}, \{b^m\}} \quad & \sum_m h_m \Big(||\Pi_m u^m - \Pi_m b^m||^2 \\ & \qquad + ||(I - \Pi_m) u^m - a^m||^2 \Big) \\ s.t. \quad & D_m \phi = w_m D_m b^m, \quad (I - \Pi_m) a^m = a^m, \\ & \Pi_m b^m = b^m, \quad \text{for all } m \in \mathcal{M}. \end{aligned}$$

Since $I - \Pi_m$ is a projection operator, it follows that $(I - \Pi_m)(I - \Pi_m) u^m = (I - \Pi_m) u^m$. Thus, for all $m \in \mathcal{M}$, $a^m = (I - \Pi_m) u^m$ is feasible in the above optimization problem. Additionally, it can be seen that this choice of a^m is the unique minimizer of the cost for all b^m. Therefore, setting $a^m = (I - \Pi_m) u^m$ for all $m \in \mathcal{M}$ the above optimization problem can be reduced to

$$\begin{aligned} \min_{\phi, \{b^m\}} \quad & \sum_m h_m ||\Pi_m u^m - \Pi_m b^m||^2 \\ s.t. \quad & D_m \phi = w_m D_m b^m, \quad \Pi_m b^m = b^m, \\ & \text{for all } m \in \mathcal{M}. \end{aligned} \tag{1}$$

Using the definition of the norm and the identities $h_m\Pi_m = \Delta_{0,m} = D_m^* D_m$, $D_m\Pi_m = D_m$, it follows that

$$\begin{aligned}
h_m||\Pi_m u^m - \Pi_m b^m||^2 \\
&= h_m\langle \Pi_m(u^m - b^m), \Pi_m(u^m - b^m)\rangle \\
&= \langle \Pi_m(u^m - b^m), (h_m\Pi_m)(u^m - b^m)\rangle \\
&= \langle \Pi_m(u^m - b^m), \Delta_{0,m}(u^m - b^m)\rangle \\
&= \langle \Pi_m(u^m - b^m), D_m^* D_m(u^m - b^m)\rangle \\
&= \langle D_m\Pi_m(u^m - b^m), D_m(u^m - b^m)\rangle \\
&= \langle D_m(u^m - b^m), D_m(u^m - b^m)\rangle.
\end{aligned}$$

Therefore, the optimization problem in (1) is equivalent to,

$$\begin{aligned}
\min_{\phi,\{b^m\}} \quad & \langle D_m(u^m - b^m), D_m(u^m - b^m)\rangle \\
s.t. \quad & D_m\phi = w_m D_m b^m, \quad \Pi_m b^m = b^m, \\
& \text{for all } m \in \mathcal{M}.
\end{aligned} \tag{2}$$

Note that since $D_m\Pi_m = D_m$ and Π_m is a projection operator, for any given ϕ, $b^m = \frac{1}{w_m}\Pi_m\phi$ gives a feasible solution of the above optimization problem. Moreover, for a given ϕ, this b_m is unique, since if there are two b^m functions satisfying $D_m\phi = w_m D_m b^m$, then their difference belongs to the kernel of D_m, hence for at least one of these functions the constraint $\Pi_m b^m = b^m$ cannot be met. Therefore, substituting $b^m = \frac{1}{w_m}\Pi_m\phi$, and using the identity $D_m\Pi_m = D_m$, the optimization problem in (2) can be turned into the following unconstrained optimization problem:

$$\min_{\phi} \quad \sum_m \langle D_m u^m - \frac{1}{w_m}D_m\phi, D_m u^m - \frac{1}{w_m}D_m\phi\rangle.$$

The normal equation of the above optimization problem suggests that the optimal ϕ is a solution of

$$\sum_m \frac{1}{w_m^2}\Delta_{0,m}\phi = \sum_m \frac{1}{w_m}\Delta_{0,m}u^m.$$

Since, it is equivalent to the optimal objective value of a norm minimization problem, the optimal solution in the unconstrained optimization problem is bounded. Thus, there exists a ϕ satisfying the normal equation, and one such ϕ is given by[1]

$$\phi = \left(\sum_m \frac{1}{w_m^2}\Delta_{0,m}\right)^{\dagger} \sum_m \frac{1}{w_m}\Delta_{0,m}u^m. \tag{3}$$

Therefore, for every $m \in \mathcal{M}$ the solution of P1 is given by

$$\hat{u}^m = a^m + b^m = (I - \Pi_m)u^m + \frac{1}{w_m}\Pi_m\phi,$$

where ϕ is given in (3). ■

If all weights are equal to 1, the above theorem provides a closed form solution for projection to the set of exact potential games.

[1]Since $\Delta_{0,m} = h_m\Pi_m$, it follows that the kernel of $\sum_m \frac{1}{w_m^2}\Delta_{0,m}$, is in the intersection of kernels of the projection operators $\{\Pi_m\}$. This space is spanned by functions that equal to a constant at every point of their domains. Hence, the potential is unique up to a constant addition. Note that the difference between any two solutions of ϕ belongs to the kernel of $\Pi_m = D_m^\dagger D_m$ [8].

B. *Choosing Weights for the Potential Game Approximation*

In this section we discuss methods for choosing weights to obtain a close weighted potential game to a given game. As shown in Section III, $\mathcal{WP}$ is nonconvex, implying that finding the closest weighted potential game to an arbitrary game involves solving nonconvex optimization problems. In particular, for any given game, the solution of problem P1 provides the closest fixed-weight potential game. The best weights (and the closest potential game to the original game) can be found by minimizing the optimal objective value of P1 over all weights. Using the closed form solution of P1, its optimal objective value as a function of the weights can be given as:

$$\sum_m h_m \left|\left|\Pi_m u^m - \Pi_m \left(\sum_k \frac{1}{w_k^2}\Delta_{0,k}\right)^{\dagger} \sum_k \frac{1}{w_k}\Delta_{0,k}u^k\right|\right|^2.$$

This is a nonconvex function of the weights, and hence solving for the optimal weights requires minimizing the above nonconvex function over the set of possible weights.

We next consider an alternative formulation for finding a close weighted potential game. For a given set of weights $\{w_m\}$, we first consider the problem

$$(\text{P2 :}) \quad \begin{aligned}
\min_{\phi,\{\hat{u}^m\}} \quad & \sum_m h_m||w_m u^m - w_m\hat{u}^m||^2 \\
s.t. \quad & D_m\phi = w_m D_m\hat{u}^m, \\
& \text{for all } m \in \mathcal{M}.
\end{aligned}$$

It can be seen that P2 is a convex optimization problem, and it gives a weighted potential game approximation of the original game. To obtain a good approximation, we are interested in solving P2 with the weights that minimize its optimal objective value. Solving P2 with the best weights is equivalent to introducing a new variable $\hat{v}^m = w_m\hat{u}^m$ and solving the following convex optimization problem:

$$\begin{aligned}
\min_{\phi,\{w_m\},\{\hat{v}^m\}} \quad & \sum_m h_m||w_m u^m - \hat{v}^m||^2 \\
s.t. \quad & D_m\phi = D_m\hat{v}^m, \\
& \text{for all } m \in \mathcal{M}.
\end{aligned}$$

Using the solution of this problem, the optimal utilities corresponding to the best weights in P2 can be recovered from $\hat{u}^m = \frac{1}{w_m}\hat{v}^m$. Note that in the new formulation the optimal weights and the utilities can be obtained by solving a convex optimization problem.

We next obtain a closed form solution for P2. Note that the optimization formulations P1 and P2 are related: Consider a set of utility functions $\{u^m\}$ and assume that for a fixed set of weights $\{w_m\}$, the solution of P2 is given by $\{\hat{u}^m\}$ and ϕ. Comparing the optimization formulations P1 and P2, it can be seen that the closest exact potential game, in the sense of P1, to the game with utility functions $\{w_m u^m\}$ is given by $\{w_m\hat{u}^m\}$ and this game also has a potential function ϕ. Hence, the solution of P2 can be obtained by: (i) scaling the utility functions of the original game by weights $\{w_m\}$, (ii) solving P1 with the scaled utility functions and unit weights, and (iii) scaling the utility functions that solve P1 by weights $\left\{\frac{1}{w_m}\right\}$. The following theorem, uses this observation to characterize the optimal solution of P2.

Theorem 4: The optimal solution of P2 is given by $\phi = \Delta_0^\dagger \sum_m \Delta_{0,m} w_m u^m$ and $\hat{u}^m = \frac{1}{w_m}\Pi_m \phi + (I - \Pi_m)u^m$ for all players $m \in \mathcal{M}$.

Proof: In order to prove the claim we use the fact that optimal solution of P2, denoted by $\{\hat{u}^m\}$, is such that $\{w_m \hat{u}^m\}$ is the closest exact potential game to the game with utilities $\{w_m u^m\}$. Setting all weights in P1 to 1, and using Theorem 3 it follows that the closest exact potential game to the game with payoff functions $\{w_m u^m\}$ has a potential function $\phi = (\sum_m \Delta_{0,m})^\dagger \sum_m \Delta_{0,m}(w_m u^m)$ and the optimal payoff functions $\{w_m \hat{u}^m\}$ are given by

$$w_m \hat{u}^m = \Pi_m \phi + (I - \Pi_m) w_m u^m.$$

Substituting ϕ to the above equation it follows that

$$\hat{u}^m = \frac{1}{w_m}\Pi_m \left(\sum_m \Delta_{0,m}\right)^\dagger \sum_k \Delta_{0,k} w_k u^k + (I - \Pi)u^m.$$

The claim follows since $\Delta_0 = \sum_m \Delta_{0,m}$. ■

Using this theorem, it follows that for a fixed set of weights, the optimal cost of P2 is

$$\psi(w) \triangleq \sum_m h_m \left|\left| w_m \Pi_m u^m - \Pi_m \Delta_0^\dagger \sum_m \Delta_{0,m} w_m u^m \right|\right|^2,$$

where w denotes the vector of all weights $\{w_m\}$. Note that ψ is a convex function of the weights. Hence, weights that will lead to a better potential game approximation can be obtained by solving

$$\min_w \quad \psi(w) \qquad s.t. \;\; w_m \geq 1 \qquad \text{for all } m \in \mathcal{M}. \tag{4}$$

We refer to the solution of P2, using the optimal weights with respect to (4), as the *best weighted potential game approximation in the sense of P2*. If the original game is a weighted potential game, it can be seen that the best weighted potential game approximation of this game is itself.

Proposition 5: Let $\mathcal{G}$ be a weighted potential game. The best weighted potential game approximation of $\mathcal{G}$, in the sense of P2, is itself.

Proof: Without loss of generality, assume that $\mathcal{G}$ is a weighted potential game with weights $w_m \geq 1$ for all $m \in \mathcal{M}$. Observe that the optimal objective value of P2, for these weights is equal to 0. Since it minimizes ψ, the best weighted potential game approximation of the original game in the sense of P2, also results in zero objective value in P2. Note that since the objective function of P2 is a weighted norm for any set of strictly positive weights, it is equal to 0, only when $u^m = \hat{u}^m$ for all $m \in \mathcal{M}$. Thus, it follows that the best weighted potential game approximation of $\mathcal{G}$, in the sense of P2 is unique and it is equal to $\mathcal{G}$. ■

Note that the above proposition also provides a way of checking whether a game is a weighted potential game or not: Given a game if the optimal objective value of (4) is equal to zero, then the original game is a weighted potential game, with weights that achive this objective value.

We conclude this section by presenting an alternative motivation for the formulation P2. Note that the strategic considerations in a game (such as the equilibria, and the best responses) do not change if utility functions in the game are scaled by different positive scalars. Hence, it makes sense to consider projections of scaled utility functions (with different weights) to the set of exact potential games, as formalized through problem P2. This approach provides a higher degree of freedom and leads potentially to a smaller distance than the distance from the closest exact potential game. We show in Section VI that tighter approximations of both static and dynamic properties of a game can be obtained through this approach.

C. Ordinal Potential Games

As shown in Section III, the set of ordinal potential games is a nonconvex subset of $\mathcal{G}_{\mathcal{M},E}$. Because of this, finding the closest ordinal potential game to a given game also requires solving a nonconvex optimization problem. However, it is possible to develop a convex optimization formulation, similar to the one we proposed for weighted potential games, in order to find a close ordinal potential game. The following alternative characterization of ordinal potential games can be used for this. The proof is immediate from the definition of ordinal potential games and is omitted.

Lemma 2: A game is an ordinal potential game if and only if there exist functions $\lambda_m : E \times E \to \mathbb{R}$ such that $\lambda_m(\mathbf{p}, \mathbf{q}) \geq 1$ for all $(\mathbf{p}, \mathbf{q}) \in A_m$ and there exists a potential function ϕ such that $\phi(\mathbf{p}) - \phi(\mathbf{q}) = \lambda_m(\mathbf{p}, \mathbf{q})(u^m(\mathbf{p}) - u^m(\mathbf{q}))$, for all m and $(\mathbf{p}, \mathbf{q}) \in A_m$.

The functions $\{\lambda_m\}$ in the above lemma are similar to the weights in the case of weighted potential games. The difference is that for ordinal potential games the weights not only depend on the identity of the players, but they are also functions of the strategy profiles.

This alternative characterization suggests that for fixed functions $\{\lambda_m\}$ the closest ordinal potential game can be obtained by a convex optimization problem. Search over optimal weights requires a nonconvex optimization formulation, but analogous to P2 and (4), a related convex optimization problem which yields a close ordinal potential game can be obtained. The details are omitted due to page constraints.

V. Example

In this section, we compare the methods for finding close potential games proposed in the previous section on a specific example.

We focus on two-player games where each player has three strategies. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

For any $\alpha \in [0, 1]$, let $\mathcal{G}_\alpha$ denote the game with payoff matrices $(u^1, u^2) = (\alpha \cdot 3A + (1-\alpha) \cdot B, \alpha \cdot A + (1-\alpha) \cdot 3B)$.

It can be seen from this definition that $\mathcal{G}_0$ is a weighted potential game with payoff matrices $(u^1, u^2) = (B, 3B)$ and weights $(3, 1)$ and similarly $\mathcal{G}_1$ is a weighted potential game with payoff matrices $(u^1, u^2) = (3A, A)$ and weights $(1, 3)$.

In our simulations, for different values of α, we consider the game $\mathcal{G}_\alpha$ and compute (i) the closest exact potential game (ii) the best weighted potential game approximation using the formulation P2 (iii) the closest weighted potential game to $\mathcal{G}_\alpha$. Here, we obtain the closest weighted potential game to $\mathcal{G}_\alpha$ by exhaustive search over weights (w_1, w_2), and the corresponding solution of P1. We calculate the distance from the original game to each of these games, by quantifying the distance between the games in terms of the norm defined in the space of games (for $\mathcal{G}$ with payoffs $\{u^m\}$, $||\mathcal{G}||^2 = \sum_m h_m ||u^m||^2$).

In Figure 1 we compare the distances between the original game and the close potential games obtained by different methods, as a function of α. We see that the best weighted potential game approximation in the sense of P2, closely approximates the closest weighted potential game, and it improves significantly over the closest exact potential game.

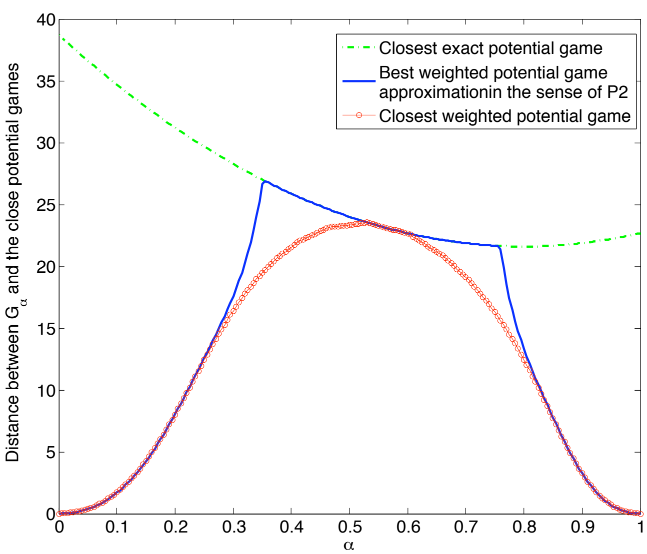

Fig. 1. Comparison of distances between the original game and the close potential games obtained by different methods.

VI. Static and Dynamic Properties of Near-Potential Games

In this section we relate the static and dynamic properties of a given game with those of a nearby potential game.

A. Static Properties

If two games have similar payoff functions, then their approximate equilibria are closely related. In this section, we use this property to study their sets of equilibria. The following lemma characterizes the ϵ-equilibria of a game in terms of the ϵ-equilibria of a nearby game.

Lemma 3: Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be two games with set of players $\mathcal{M}$, strategy profiles E and utility functions $\{u^m\}$ and $\{\hat{u}^m\}$ respectively. Assume that $|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \epsilon_0$ for every $m \in \mathcal{M}$ and $\mathbf{p} \in E$. Then,

(i) For any two strategy profiles $\mathbf{p}$, and $\mathbf{q}$ that only differ in the strategy of player k, it follows that $u^k(\mathbf{q}) - u^k(\mathbf{p}) \leq \hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p}) + 2\epsilon_0$.

(ii) Every ϵ_1-equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq 2\epsilon_0 + \epsilon_1$.

Proof: (i) Let $\mathbf{p}$, and $\mathbf{q}$ be two strategy profiles that differ only in the strategy of player k, and let $c = \hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p})$. Since $|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \epsilon_0$, for every $m \in \mathcal{M}$ and $\mathbf{p} \in E$, it follows that

$$\begin{aligned} u^k(\mathbf{q}) - u^k(\mathbf{p}) &= u^k(\mathbf{q}) - u^k(\mathbf{p}) - (\hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p})) + c \\ &\leq 2\epsilon_0 + c, \end{aligned}$$

as the claim suggests.

(ii) If $\mathbf{p}$ is an ϵ_1-equilibrium of $\hat{\mathcal{G}}$, then $\hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p}) \leq \epsilon_1$ for every $\mathbf{q}$ that differs from $\mathbf{p}$ only in the strategy of player k. From part (i) it follows that $u^k(\mathbf{q}) - u^k(\mathbf{p}) \leq 2\epsilon_0 + \epsilon_1$. Since this inequality holds for every $\mathbf{q}$ that differs from $\mathbf{p}$ in the strategy of only player k and k is arbitrary, it follows that $\mathbf{p}$ is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq 2\epsilon_0 + \epsilon_1$. ■

The previous lemma can be used to characterize the approximate equilibrium set of an arbitrary game in terms of the approximate equilibrium set of a close potential game.

Proposition 6: Let a game $\mathcal{G}$ and a set of weights $\{w_m\}$ be given. For $\mathcal{G}$ and these weights, denote the game which solves P2 by $\hat{\mathcal{G}}$ and the corresponding objective value in P2 by $\psi(w) = \alpha^2$. Assume that h_m denotes the number of strategies player m has. Then, every ϵ_1-equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq \epsilon_1 + 2\max_m \frac{\alpha}{w_m\sqrt{h_m}}$.

Proof: Denote the utilities of $\mathcal{G}$ and $\hat{\mathcal{G}}$ by $\{u^m\}_{m\in\mathcal{M}}$ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ respectively. Consider any strategy profile $\mathbf{p}$. By definition of α it follows that

$$h_k |w_k u^k(\mathbf{p}) - w_k \hat{u}^k(\mathbf{p})|^2 \leq \alpha^2,$$

for all $k \in \mathcal{M}$, $\mathbf{p} \in E$. Thus, it follows that $|u^k(\mathbf{p}) - \hat{u}^k(\mathbf{p})| \leq \frac{\alpha}{w_k\sqrt{h_k}} \leq \max_m \frac{\alpha}{w_m\sqrt{h_m}}$ for all k and $\mathbf{p} \in E$. Therefore, Lemma 3 implies that any ϵ_1 equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq \epsilon_1 + 2\max_m \frac{\alpha}{w_m\sqrt{h_m}}$. ■

B. Dynamic Properties

We next show that dynamics in an arbitrary game can be studied using a close potential game. Before stating our result, we introduce the approximate better-response dynamics:

Definition 6.1 (ϵ-Better-Response Dynamics): The updates take place in a round-robin manner and at any update a single user can modify its strategy. Let player m be the player chosen for updating its strategy. If there exist strategies which improve its payoff by more than $\epsilon > 0$, player m updates its strategy to one such strategy, otherwise it does not modify its strategy.

The following proposition shows that in an arbitrary game, approximate convergence of the better-response dynamics can be studied using the properties of a close potential game.

Proposition 7: Let a game $\mathcal{G}$ and a set of weights $\{w_m\}$ be given. For $\mathcal{G}$ and these weights, denote the game which solves P2 by $\hat{\mathcal{G}}$ and the corresponding objective value in P2 by $\psi(w) = \alpha^2$. Assume that $\epsilon \geq 2\max_m \frac{\alpha}{w_m\sqrt{h_m}}$. In $\mathcal{G}$, the ϵ-approximate better-response dynamics is confined after finite number of round-robin iterations in the ϵ-equilibrium set.

Proof: If the ϵ-equilibrium set is reached, by the definition of the dynamics, no player modifies its strategy. Therefore, to show that the dynamics is confined in the ϵ-equilibrium set, it is sufficient to show that dynamics reach to this set starting from an arbitrary strategy profile.

Let $\{u^m\}$ and $\{\hat{u}^m\}$ be the utility functions in $\mathcal{G}$ and $\hat{\mathcal{G}}$ respectively. From the definition of α it follows that

$$|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \frac{\alpha}{w_m\sqrt{h_m}} \leq \max_k \frac{\alpha}{w_k\sqrt{h_k}}, \qquad (5)$$

for any strategy profile $\mathbf{p}$. When player m updates its strategy, its payoff increases by more than ϵ. Thus, Lemma 3 (i) and (5) imply that in such an update, $\hat{u}^m$ increases by more than $\epsilon - 2\max_m \frac{\alpha}{w_m\sqrt{h_m}} \geq 0$. Since $\hat{\mathcal{G}}$ is a potential game, this implies that the potential function of $\hat{\mathcal{G}}$ strictly increases at each update. Consequently, no strategy profile can be visited twice by this update process. Since there are finitely many strategy profiles, the update process has to terminate. Hence, the ϵ-equilibrium set is not empty and it is reached in finitely many updates. Moreover, if an ϵ-equilibrium is not reached, then at each round-robin iteration a player who can improve its payoff is found. Therefore, in finitely many round-robin iterations, the dynamics reaches to the ϵ-equilibrium set. ■

Remarks: 1) The above proposition implies that if $\mathcal{G}$ is a weighted potential game, i.e., if $\alpha = 0$, then with $\epsilon = 0$, the better-response dynamics converges to a Nash equilibrium in $\mathcal{G}$.

2) The assumption that the updates take place in a round-robin order is present to simplify the proof. If instead of round-robin updates, at each time instant some of the agents were randomly chosen to update their strategy, the update process would still converge to the ϵ-equilibrium set. In this case, it is also possible to characterize the expected time to converge to the ϵ-equilibrium set.

3) Using a close potential game, it can be proved that the best response dynamics, i.e., dynamics in which each player who updates its action chooses a strategy that maximizes its payoff, also converges to a set of approximate equilibria. Therefore, in the above proposition, the condition of choosing an approximate payoff maximizer can be relaxed. Similarly, if continuous time dynamics, such as the smoothened fictitious play, are used, convergence to an ϵ-equilibrium set can still be established. See [9] for details.

Note that in Propositions 6 and 7, $\alpha = \sqrt{\psi(w)}$ is a function of the weights $\{w_m\}$. As discussed in Section IV, the best weighted potential game approximation in the sense of P2 achieves the minimum of α over all weights $w_m \geq 1$, thus it leads to a smaller α and larger weights, when compared to the optimal solution of P2 for weights $w_m = 1$. Note that the optimal solution of the latter is the closest exact potential game to the original game. Hence, the bounds in Propositions 6 and 7 can be improved using the best weighted potential game approximation of the original game, instead of the closest exact potential game to it. Therefore, the best weighted potential game approximation of a game can be used to obtain a tighter characterization of the static and dynamic properties of the original game.

VII. Conclusions

We introduced a geometric framework for the analysis of arbitrary games in terms of "close" potential games. We showed that the sets of weighted and ordinal potential games are nonconvex, as opposed to fixed-weight potential games, which form a subspace. Using this fact, we obtained closed-form solutions for the closest exact and fixed-weight potential games to a given game. Additionally, we introduced a convex optimization formulation which provides a close weighted (and ordinal) potential game with arbitrary weights to any given strategic-form finite game.

Our results show that the static and dynamic properties of an arbitrary game can be analyzed by employing the properties of potential games that are close to it. Moreover, the proposed scheme for finding a close weighted potential game results in a tighter characterization of static and dynamic properties of a game, when compared to the results that can be obtained using exact potential games. We leave the application of our potential game approximation framework to analysis of various update rules and additional static properties, such as efficiency notions, as a future goal.

References

[1] D. Monderer and L. Shapley, "Potential games," *Games and economic behavior*, vol. 14, no. 1, pp. 124–143, 1996.

[2] ——, "Fictitious play property for games with identical interests," *Journal of Economic Theory*, vol. 68, no. 1, pp. 258–265, 1996.

[3] J. Hofbauer and W. Sandholm, "On the global convergence of stochastic fictitious play," *Econometrica*, pp. 2265–2294, 2002.

[4] J. Marden, G. Arslan, and J. Shamma, "Cooperative Control and Potential Games," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 39, no. 6, pp. 1393–1407, 2009.

[5] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, "Near-Optimal Power Control in Wireless Networks: A Potential Game Approach ," *INFOCOM 2010. The 29th Conference on Computer Communications. IEEE*, 2010.

[6] G. Arslan, J. Marden, and J. Shamma, "Autonomous vehicle-target assignment: A game-theoretical formulation," *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, p. 584, 2007.

[7] M. Voorneveld and H. Norde, "A Characterization of Ordinal Potential Games," *Games and Economic Behavior*, vol. 19, no. 2, pp. 235–242, 1997.

[8] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, "Flow representations of games: Harmonic and potential games," LIDS, MIT, Tech. Rep., 2010.

[9] ——, "Dynamics in games and near potential games," LIDS, MIT, Tech. Rep., 2010, working Paper.