

Stock Repurchase with an Adaptive Reservation Price: A Study of the Greedy Policy

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Abstract. We consider the problem of stock repurchase over a finite time horizon. We assume that a firm has a reservation price for the stock, which is the highest price that the firm is willing to pay to repurchase its own stock. We characterize the optimal policy for the trader to maximize the total number of shares he can buy over a fixed time horizon. In particular, we study a greedy policy, which involves in each period buying a quantity that drives stock price to the reservation price.

Key words: stock repurchase, dynamic programming, reservation price.

1 Introduction

Since 1980, there has been an extraordinary growth in the use of stock (share) repurchases in the world's major financial markets. Grullon and Michaely [8] report that in the U.S., expenditures on stock repurchase programs (relative to total earnings) increased from 4.8 percent in 1980 to 41.8 percent in 2000. In 2005, the use of share repurchases had grown to \$ 349 billion in the U.S. while it was \$5 billion in 1980. In the financial markets of UK and Japan, hundreds of firms have made stock repurchase announcements in the last two decades. Today, stock repurchase is becoming a popular financial strategy among firms. There are many incentives for firms to buy back their own stock in a share repurchase. The major motivations are to take advantage of potential undervaluation, distribute excess capital, alter their leverage ratio, fend off takeovers and counter the dilution effects of stock options. For an excellent study of motivations for stock repurchases, we refer the reader to Dittmar[7].

Here, we give an example of stock repurchase that specifies in advance a single purchase price, the number of shares sought and the duration of the offer. TDK corporation (TYO:6762, NYSE:TDK, LSE:TDK) is a leading Japanese company that manufactures electronic materials, electronic components, recording and data-storage media. On May 15, 2007, the corporation's board of directors made a decision to buy back up to 4 million shares of TDK corporation common stock with the total cost up to 44 billion yen from May 16, 2007 through June 30, 2007. On May 29, 2007, TDK

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announced that the company would place an order to purchase shares at 8:45am on May 30, 2007 through Tokyo stock exchange trading network system. The price in the purchase order would be 10900 yen per share. The number of shares to be purchased would be up to 4 million shares. In the announcement, the company also noted that market conditions may mean that some or all the shares are not purchased. On July 2, 2007 (two days after the end day of share repurchase period), TDK corporation announced the results of share repurchase and completion of the share repurchase program. The corporation purchased 3.599 million shares with a total cost of 39.2291 billion yen through Tokyo stock exchange trading network system on May 30, 2007. From May 31 to June 30, 2007, TDK corporation repurchased 0 share of its own stock.

Due to the large trading volume (up to 4 million shares in the TDK example), significant impact on the stock price is unavoidable. In the last few years, several studies have been done on dynamic optimal trading strategies that minimize the expected cost of trading a large block of stock. Specifically, suppose that a trader has to buy Q units of a stock over $N + 1$ periods. Let q_i denote the trader's order size for the stock at period i . Then, this problem can be expressed as:

$$\min_{q_i} E\{\sum_{i=0}^N p_i q_i\} \quad (1)$$

$$s.t. \quad \sum_{i=0}^N q_i = Q. \quad (2)$$

In each period, the price of the stock is a function of the trader's order size. Evolution of the price p_i may be expressed as

$$p_i = p_{i-1} + \theta q_i + \epsilon_i, \quad (3)$$

where θ is a positive constant and ϵ_i is a random variable that represents the price change made by the market. This model first appeared in Bertsimas and Lo [6]. They show that to minimize expected execution cost, a trader should split his order evenly over time. Almgren and Chriss [5], Huberman and Stanzl [9] and Schied, Schöneborn and Tehranchi[11] extend the Bertsimas and Lo framework to allow risk aversion and temporary price impact. Alfonsi, Schied and Schulz [1] model the dynamics of supply/demand in a limit-order-book market. We also refer the reader to Alfonsi and Schied[3], Alfonsi, Schied and Schulz [2] and Almgren [4] for nonlinear price impact model and Moallemi, Park and Van Roy [10] and Schöneborn and Schied[12] for trading in a competitive setting.

(1)-(2) is a good model if the trader has an obligation to buy Q units. However, in the case of stock repurchase like the TDK example, the trader does not have this obligation. Although the firm has a target of purchasing up to 4 million shares, the repurchase program was completed with actually 3.599 million shares. Instead, the trader has a reservation price, 10900 yen per share, which is the highest price the firm is willing to pay for the stock. Therefore, the trader's objective is to maximize the total number of shares (with share cap and budget cap, 4 million shares and 44 billion yen in the TDK example) he can buy in the $N + 1$ periods below a reservation price \bar{p} , ($\bar{p} = 10900$ yen in the TDK example). If, at some time period, the trader runs into the share cap or the budget cap, then he can stop. If, at the end of the last period, the target number of shares is still unmet, the trader at least has bought as many units as possible below the reservation price \bar{p} . This is what happened in the TDK example.

In the open market stock repurchase, a firm may or may not announce that it will repurchase some shares in the open market from time to time as market conditions dictate and maintains the

option of deciding whether, when and how many shares to buy at what price. In our model, we give the trader the full flexibility to adjust his reservation price from \bar{p}_i to \bar{p}_{i+1} as the market conditions change, i.e., at the beginning of period i , the trader only needs to know the reservation price for the current period, \bar{p}_i , and doesn't need to know \bar{p}_j for $j > i$. In many papers of the literature, $\{\epsilon_i\}_{i=0}^N$ that represent the market conditions are assumed to be i.i.d with zero mean. As noted in Huberman and Stanzl [9], the zero-mean assumptions are not made for convenience. In this paper, we relax these assumptions. Actually, we don't make any assumption on $\{\epsilon_i\}_{i=0}^N$.

In this paper, we use the structure developed in Almgren and Chriss [5] to model how stock's price is affected by the trader's order and its evolution over periods. This model is also used in Huberman and Stanzl [9]. In particular, in each period, the trader's order has temporary and permanent impact on the stock's price. The initial price of the stock at time i , p_i , is observed by the trader. Given this price, the trader faces the transaction price $\hat{p}_i = p_i + \lambda_{1i}q_i$ to buy the quantity q_i , where λ_{1i} is a positive constant that measures the temporary price impact of the trader's order. The new initial price for the next period evolves according to $p_{i+1} = p_i + \lambda_{2i}q_i + \epsilon_i$, where λ_{2i} is a positive constant that measures the permanent price impact of the trader's order and ϵ_i is a random variable. Given this law of motion for price p_i and the above analysis, the optimal policy of the trader is given as the optimal solution of the following optimization problem:

$$\text{maximize}_{q_i} \quad E\{\sum_{i=0}^N q_i\} \quad (4)$$

$$\text{subject to} \quad \hat{p}_i = p_i + \lambda_{1i}q_i \quad (5)$$

$$p_{i+1} = p_i + \lambda_{2i}q_i + \epsilon_i \quad (6)$$

$$\hat{p}_i \leq \bar{p}_i, \quad i = 0, \dots, N. \quad (7)$$

The objective of this paper is to characterize the optimal trading strategy based on the above formulation. It is easy to see that if $p_i \geq \bar{p}_i$ in period i , the trader will not purchase anything in this period. On the other hand, it is not clear how much should the trader purchase when $p_i < \bar{p}_i$. It is tempting to conclude that since the trader's goal is to maximize the number of units of stock, the trader should purchase to increase price up to \bar{p}_i , i.e., purchase $q_i^* = \frac{\bar{p}_i - p_i}{\lambda_{1i}}$. This is the strategy used in the TDK example. We refer to this policy as the *greedy policy*. Unfortunately, we show in Section 3.1, using a counter example, that the greedy policy is not always optimal. The following questions are therefore natural: What is the structure of the optimal policy? Under what conditions the greedy policy is optimal? And, when it is not optimal, how far is it from the optimal?

In Section 2, we show that in each period, the optimal policy of (4)-(7) is either the greedy policy or a no-buy policy. In Section 3.2, we show that under some reasonable conditions, the greedy policy is indeed optimal. Remarkably, we need only impose conditions on the price impact parameters λ_{ki} , $k = 1, 2$ and $i = 0, \dots, N$. We do not need to make any assumption on the random variable ϵ_i , price p_i or reservation price \bar{p}_i . In Section 3.3, we study the performance of the greedy policy when it is not optimal. We derive a lower bound on the ratio of the value returned by the greedy policy to the optimal value of (4)-(7).

2 The Optimal Policy

In this section, we show that in each period, the optimal policy of (4)-(7) is either the greedy policy or a no-buy policy. We need the following lemmas.

Lemma 2.1. *If a function $f(x)$ is convex over $[a, b]$, then $\max_{x \in [a, b]} f(x) = \max\{f(a), f(b)\}$.*

This lemma is straight-forward, which tells us that the maximal value of a univariate convex function is achieved at one of the endpoints.

Lemma 2.2. *Assume $f_1(x)$ and $f_2(x)$ are convex functions. Given any z , define $f(x) = \max\{f_1(x), f_2(x)\}$ for $x \leq z$, and $f(x) = f_1(x)$ for $x \geq z$. Then $f(x)$ is a convex function.*

Proof. Assume $x \leq z' \leq y$. If $z' \geq z$,

$$f(z') = f_1(z') \leq \frac{y-z'}{y-x} f_1(x) + \frac{z'-x}{y-x} f_1(y) \leq \frac{y-z'}{y-x} f(x) + \frac{z'-x}{y-x} f(y), \quad (8)$$

where the first inequality follows from the convexity of $f_1(x)$ and the second inequality follows from the definition of $f(x)$.

If $z' \leq z$, we only need to consider the case of $x \leq z' \leq z \leq y$ since the case of $x \leq z' \leq y \leq z$ is trivial. We have

$$f(z') \leq \frac{z-z'}{z-x} f(x) + \frac{z'-x}{z-x} f(z) \leq \frac{z-z'}{z-x} f(x) + \frac{z'-x}{z-x} \left(\frac{y-z}{y-x} f(x) + \frac{z-x}{y-x} f(y) \right) = \frac{y-z'}{y-x} f(x) + \frac{z'-x}{y-x} f(y), \quad (9)$$

where the first inequality follows that $f(x)$ is convex over $(-\infty, z]$ and the second inequality follows from (8). \square

For problem (4)-(7), we define $J_i(p_i)$ to be the optimal value to go from period i at price p_i . For $i = N$,

$$J_N(p) = \begin{cases} \frac{\bar{p}_N - p}{\lambda_{1N}} & \text{if } p < \bar{p}_N, \\ 0 & \text{if } p \geq \bar{p}_N, \end{cases} \quad (10)$$

which is a convex function. Assume $J_{i+1}(p)$ is a convex function. For any $p \geq \bar{p}_i$, $J_i(p) = E(J_{i+1}(p + \epsilon_i))$. For any $p \leq \bar{p}_i$,

$$\begin{aligned} J_i(p) &= \max_{0 \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i} q_i + \epsilon_i)) \\ &= \max\left\{ E(J_{i+1}(p + \epsilon_i)), \frac{\bar{p}_i - p}{\lambda_{1i}} + E(J_{i+1}(p + \lambda_{2i} \frac{\bar{p}_i - p}{\lambda_{1i}} + \epsilon_i)) \right\}, \end{aligned} \quad (11)$$

where the second equality follows from Lemma 2.1. Lemma 2.2 implies that $J_i(p)$ is a convex function. Hence, Lemma 2.1 implies the following optimal policy of (4)-(7).

Theorem 2.1. *In period i , if the stating price $p_i < \bar{p}_i$, the optimal order quantity q_i^* is either 0 or $\frac{\bar{p}_i - p_i}{\lambda_{1i}}$.*

3 The Greedy Policy

In this section, we show that the greedy policy is not always optimal, and give conditions under which the greedy policy is optimal. We also derive a bound to measure the performance of the greedy policy.

3.1 Greedy Policy is Not Always Optimal

Example 3.1. We consider a two-period model, $N=1$, and assume $\bar{p}_0 = \bar{p}_1 = \bar{p}$. Theorem 2.1 implies that if $J_0(p_0)$ is achieved at $E(J_1(p_0 + \epsilon_0))$ the trader should not order anything in period 0; otherwise he should use the greedy policy in that period.

We define $s = \bar{p} - p_0$, $a = 1 - \frac{\lambda_{20}}{\lambda_{10}}$ and

$$\begin{aligned} \Delta &= E(J_1(p_0 + \epsilon_0)) - E(J_1(p_0 + \lambda_{20} \frac{\bar{p} - p_0}{\lambda_{10}} + \epsilon_0)) - \frac{\bar{p} - p_0}{\lambda_{10}} \\ &= \int_{-\infty}^{\bar{p}-p_0} \frac{\bar{p} - p_0 - t}{\lambda_{11}} \phi(t) dt - \int_{-\infty}^{\bar{p}-p_0 - \lambda_{20} \frac{\bar{p}-p_0}{\lambda_{10}}} \frac{\bar{p} - p_0 - \lambda_{20} \frac{\bar{p}-p_0}{\lambda_{10}} - t}{\lambda_{11}} \phi(t) dt - \frac{\bar{p} - p_0}{\lambda_{10}} \\ &= \frac{1}{\lambda_{11}} \left(\int_{-\infty}^s (s - t) \phi(t) dt - \int_{-\infty}^{as} (as - t) \phi(t) dt \right) - \frac{s}{\lambda_{10}} = \frac{1}{\lambda_{11}} \int_{as}^s \Phi(t) dt - \frac{s}{\lambda_{10}}, \quad (12) \end{aligned}$$

where $\phi(t)$ and $\Phi(t)$ are pdf and cdf of random variable ϵ_0 . The last equality follows from integration by parts.

We set $\lambda_{11} = 0.001$, $\lambda_{10} = 0.00105$, $\lambda_{20} = 0.00315$, and assume that ϵ_0 follows normal distribution $N(0.15, 1)$. In Figure 1, the x -axis is s , the difference between the current price and the reservation price, and the y -axis is Δ , the difference between the objective function for "no-buy" policy and "greedy" policy. As you can see, there are two separate regions where the no-buy policy is optimal; otherwise the greedy is optimal.

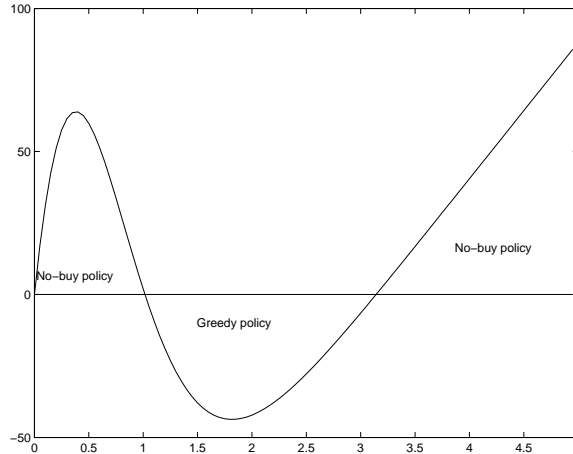


Figure 1: Optimal regions for No-buy and greedy policy.

3.2 When is Greedy Policy Optimal?

In this section, we establish conditions under which the greedy policy is an optimal solution to (4)-(7). For this purpose, we first show some properties of $J_i(p)$. For notational convenience, in this section we omit the time index of price p . The following lemma shows a monotonicity property of $J_i(p)$.

Lemma 3.1. For any $i = 0, 1, \dots, N$, $J_i(p)$ decreases with p .

Proof. We prove the lemma by applying backward induction. For $i = N$, this property easily follows from (10). Assume $J_{i+1}(p)$ decreases with p . Without loss of generality, let $p_1 \leq p_2$. We consider three cases:

Case 1: For any $p_1 \leq p_2 \leq \bar{p}_i$,

$$\begin{aligned} J_i(p_1) &= \max_{0 \leq q_i \leq \frac{\bar{p}_i - p_1}{\lambda_{1i}}} q_i + E(J_{i+1}(p_1 + \lambda_{2i}q_i + \epsilon_i)) \geq \max_{0 \leq q_i \leq \frac{\bar{p}_i - p_1}{\lambda_{1i}}} q_i + E(J_{i+1}(p_2 + \lambda_{2i}q_i + \epsilon_i)) \\ &\geq \max_{0 \leq q_i \leq \frac{\bar{p}_i - p_2}{\lambda_{1i}}} q_i + E(J_{i+1}(p_2 + \lambda_{2i}q_i + \epsilon_i)) \\ &= J_i(p_2), \end{aligned} \quad (13)$$

where the first inequality follows from the induction assumption and the second inequality follows from the facts that $\frac{\bar{p}_i - p_1}{\lambda_{1i}} \geq \frac{\bar{p}_i - p_2}{\lambda_{1i}}$ and this is a maximization problem.

Case 2: For any $p_2 \geq p_1 \geq \bar{p}_i$, we have

$$J_i(p_1) = E(J_{i+1}(p_1 + \epsilon_i)) \geq E(J_{i+1}(p_2 + \epsilon_i)) = J_i(p_2). \quad (14)$$

Case 3: For any $p_1 \leq \bar{p}_i \leq p_2$, it follows from (13) and (14) that $J_i(p_1) \geq J_i(\bar{p}_i) \geq J_i(p_2)$. \square

We now provide an upper bound on the amount lost if the starting price in period i is increased by h , i.e., an upper bound on $J_i(p) - J_i(p + h)$ for any $h > 0$ and $i = 0, \dots, N$. Consider first the last period N . For any p and $h > 0$, we must have

$$J_N(p) - J_N(p + h) \leq \frac{h}{\lambda_{1N}}. \quad (15)$$

This is easy to see by inspecting equation (10) for $p, p + h < \bar{p}_N$ or $p, p + h \geq \bar{p}_N$. If $p < \bar{p}_N$ and $p + h \geq \bar{p}_N$, then the same equation suggests that $J_N(p) - J_N(p + h) = \frac{\bar{p}_N - p}{\lambda_{1N}} \leq \frac{h}{\lambda_{1N}}$ because $p + h \geq \bar{p}_N$. The following theorem extends this observation by providing an upper bound for any i .

Theorem 3.2. Define $a_N = \frac{1}{\lambda_{1N}}$ and $a_i = \max\{a_{i+1}, \frac{1}{\lambda_{1i}}, a_{i+1} + \frac{(1 - \lambda_{2i}a_{i+1})}{\lambda_{1i}}\}$ for $i = 0, 1, \dots, N - 1$. Then for any price p and $h > 0$, we have

$$J_i(p) - J_i(p + h) \leq a_i h, \quad (16)$$

for any i .

Proof. We prove the result using backward induction. We have shown that (16) holds for $i = N$. Assume (16) holds for $i + 1$, for any p with $p + h < \bar{p}_i$, we know

$$\begin{aligned} J_i(p) &= \max_{0 \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) \\ &= \max\left\{ \max_{0 \leq q_i \leq \frac{\bar{p}_i - p - h}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)), \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) \right\}. \end{aligned}$$

We start by focusing on the first component of the right hand side of the above equation.

$$\begin{aligned} \max_{0 \leq q_i \leq \frac{\bar{p}_i - p - h}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) &\leq \max_{0 \leq q_i \leq \frac{\bar{p}_i - p - h}{\lambda_{1i}}} \{q_i + E(J_{i+1}(p + h + \lambda_{2i}q_i + \epsilon_i))\} + a_{i+1}h \\ &= J_i(p + h) + a_{i+1}h \leq J_i(p + h) + a_i h, \end{aligned} \quad (17)$$

where the first inequality follows from induction assumption, the equality follows from definition of $J_i(p + h)$, and the last inequality follows from the definition of a_i .

For the second component, observe that the definition of $J_i(p + h)$ implies that $J_i(p + h) \geq \frac{\bar{p}_i - p - h}{\lambda_{1i}} + E(J_{i+1}(p + h + \lambda_{2i}\frac{\bar{p}_i - p - h}{\lambda_{1i}} + \epsilon_i))$. It follows that

$$\begin{aligned} \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) &\leq \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} \{q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i))\} - \frac{\bar{p}_i - p - h}{\lambda_{1i}} \\ &\quad - E(J_{i+1}(p + h + \lambda_{2i}\frac{\bar{p}_i - p - h}{\lambda_{1i}} + \epsilon_i)) + J_i(p + h). \end{aligned} \quad (18)$$

If $\lambda_{2i}q_i \geq h + \lambda_{2i}\frac{\bar{p}_i - p - h}{\lambda_{1i}}$, Lemma 3.1 and inequality (18) imply that

$$\begin{aligned} \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) &\leq \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} (q_i - \frac{\bar{p}_i - p - h}{\lambda_{1i}}) + J_i(p + h) \\ &= \frac{h}{\lambda_{1i}} + J_i(p + h) \leq a_i h + J_i(p + h). \end{aligned}$$

If $\lambda_{2i}q_i < h + \lambda_{2i}\frac{\bar{p}_i - p - h}{\lambda_{1i}}$, the induction assumption and inequality (18) imply that

$$\begin{aligned} \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) &\leq \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} \{q_i - \frac{\bar{p}_i - p - h}{\lambda_{1i}}\} \\ &\quad + a_{i+1}(h + \lambda_{2i}\frac{\bar{p}_i - p - h}{\lambda_{1i}} - \lambda_{2i}q_i) + J_i(p + h) \\ &= \max_{\frac{\bar{p}_i - p - h}{\lambda_{1i}} \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} \{(1 - \lambda_{2i}a_{i+1})(q_i - \frac{\bar{p}_i - p - h}{\lambda_{1i}})\} \\ &\quad + a_{i+1}h + J_i(p + h) \\ &\leq \max\{a_{i+1}, \frac{(1 - \lambda_{2i}a_{i+1})}{\lambda_{1i}} + a_{i+1}\}h + J_i(p + h) \\ &\leq a_i h + J_i(p + h), \end{aligned} \quad (19)$$

where the second inequality follows from considering the two cases that $1 - \lambda_{2i}a_{i+1} \leq 0$ and $1 - \lambda_{2i}a_{i+1} \geq 0$.

In summary, for any $p, p + h < \bar{p}_i$, (16) holds.

We next consider any p with $p, p + h \geq \bar{p}_i$,

$$J_i(p) - J_i(p + h) = E(J_{i+1}(p + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i)) \leq a_{i+1}h \leq a_i h, \quad (20)$$

where the first inequality follows from induction assumption and the second inequality follows from definition of a_i .

The last possible case is $p < \bar{p}_i$ and $p + h \geq \bar{p}_i$. We have

$$\begin{aligned} J_i(p) - J_i(p + h) &= \max_{0 \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i)) \\ &\leq \max_{0 \leq q_i \leq \frac{h}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i)), \end{aligned} \quad (21)$$

where the last inequality follows from $\bar{p}_i - p \leq h$.

If $\lambda_{2i}q_i > h$, Lemma 3.1 and inequality (21) imply that

$$J_i(p) - J_i(p + h) \leq \max_{0 \leq q_i \leq \frac{h}{\lambda_{1i}}} q_i = \frac{h}{\lambda_{1i}} \leq a_i h \quad (22)$$

If $\lambda_{2i}q_i \leq h$, the induction assumption and inequality (21) imply that

$$\begin{aligned} J_i(p) - J_i(p + h) &\leq \max_{0 \leq q_i \leq \frac{h}{\lambda_{1i}}} q_i + a_{i+1}(h - \lambda_{2i}q_i) = \max_{0 \leq q_i \leq \frac{h}{\lambda_{1i}}} (1 - \lambda_{2i}a_{i+1})q_i + a_{i+1}h \\ &\leq \max\{a_{i+1}, \frac{(1 - \lambda_{2i}a_{i+1})}{\lambda_{1i}} + a_{i+1}\}h \leq a_i h, \end{aligned}$$

where the first inequality of second row follows from considering the two cases: $1 - \lambda_{2i}a_{i+1} \leq 0$ and $1 - \lambda_{2i}a_{i+1} \geq 0$.

In summary, for any $p < \bar{p}_i$ and $p + h \geq \bar{p}_i$, (16) holds. \square

Theorem 3.2 tells us that the difference between $J_i(p)$ and $J_i(p + h)$ is bounded by a linear function of h with slope a_i , which depends on the price impact parameters λ_{ki} but not on the price p , noise ϵ_j , $j = i, \dots, N - 1$ and how the trader sets the reservation prices \bar{p}_j in the later periods $j = i, \dots, N$. This theorem motivates the following sufficient condition under which the greedy policy is optimal.

Theorem 3.3. *At period i , if the temporary and permanent price impact parameters satisfy*

$$\lambda_{2i}a_{i+1} \leq 1, \quad (23)$$

then greedy policy is optimal to (4)-(7), i.e, if $p < \bar{p}_i$, $q_i^ = \frac{\bar{p}_i - p}{\lambda_{1i}}$ is the optimal quantity.*

Proof. For any $0 \leq q_i \leq q_i^*$, we have

$$\begin{aligned} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i)) - (q_i^* + E(J_{i+1}(p + \lambda_{2i}q_i^* + \epsilon_i))) &\leq q_i - q_i^* + a_{i+1}\lambda_{2i}(q_i^* - q_i) \\ &= (q_i - q_i^*)(1 - a_{i+1}\lambda_{2i}) \\ &\leq 0, \end{aligned} \quad (24)$$

where the first inequality follows from Theorem 3.2.

Thus, since $J_i(p) = \max_{0 \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_{2i}q_i + \epsilon_i))$, we have that $q_i^* = \frac{\bar{p}_i - p}{\lambda_{1i}}$ is the optimal quantity. \square

We now explain the intuition behind Theorem 3.3. Consider purchasing one additional unit in period i . This unit will increase the price in period $i + 1$ by λ_{2i} . Theorem 3.2 tells us that starting from period $i + 1$, we will lose at most $\lambda_{2i}a_{i+1}$ due to this price increase. Therefore, if Eq. (23) is satisfied, greedy policy is optimal since what we can lose in subsequent periods is less than what we gain in the current period.

We want to mention here that although condition (23) is independent of reservation price \bar{p}_i , it doesn't mean that reservation price has no impact on the optimality of greedy policy. Consider the trivial case that $\bar{p}_0 > 0$ and $\bar{p}_i = 0$ for all $i > 0$, then greedy policy is always optimal at period 0. Another case is that if p_j , $j > i$ is fixed and ϵ_j is sufficiently large for $j > i$, then greedy policy is optimal at period i .

Next we identify cases where the condition $\lambda_{2i}a_{i+1} \leq 1$ is satisfied, that is, we identify cases where the greedy policy is optimal. The first case is when the temporary price impact is greater than or equal to the permanent price impact, i.e., $\lambda_{1i} \geq \lambda_{2i}$, and the permanent price impact is nondecreasing with time, i.e., $\lambda_{2i} \leq \lambda_{2(i+1)}$. This is shown in the following proposition.

Proposition 3.1. *If for each i , $\lambda_{1i} \geq \lambda_{2i}$ and $\lambda_{2i} \leq \lambda_{2(i+1)}$, then the greedy policy is optimal.*

Proof. We show $\lambda_{2i}a_{i+1} \leq 1$ by backward induction. For $i = N - 1$, $\lambda_{2(N-1)}a_N = \frac{\lambda_{2(N-1)}}{\lambda_{1N}} \leq \frac{\lambda_{2N}}{\lambda_{1N}} \leq 1$. Assume $\lambda_{2i}a_{i+1} \leq 1$, then $a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} \geq a_{i+1}$. Also, $a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} = (1 - \frac{\lambda_{2i}}{\lambda_{1i}})a_{i+1} + \frac{1}{\lambda_{1i}} \geq \frac{1}{\lambda_{1i}}$, hence $a_i = (1 - \frac{\lambda_{2i}}{\lambda_{1i}})a_{i+1} + \frac{1}{\lambda_{1i}}$. Therefore, $\lambda_{2(i-1)}a_i \leq \lambda_{2i}a_i = (1 - \frac{\lambda_{2i}}{\lambda_{1i}})\lambda_{2i}a_{i+1} + \frac{\lambda_{2i}}{\lambda_{1i}} \leq (1 - \frac{\lambda_{2i}}{\lambda_{1i}}) + \frac{\lambda_{2i}}{\lambda_{1i}} \leq 1$. By Theorem 3.3, we know that the greedy policy is optimal. \square

The second case is when a unit ordered in this period will increase next period price more than current period price, i.e., $\lambda_{1i} \leq \lambda_{2i}$, but less than buying an additional unit next period, i.e., $\lambda_{2i} \leq \lambda_{1(i+1)}$. This intuitively implies that as time progresses, the stock is desirable by more and more people.

Proposition 3.2. *If for each i , $\lambda_{1i} \leq \lambda_{2i}$ and $\lambda_{2i} \leq \lambda_{1(i+1)}$, then the greedy policy is optimal.*

Proof. We show $\lambda_{2i}a_{i+1} \leq 1$ using backward induction. For $i = N - 1$, $\lambda_{2(N-1)}a_N = \frac{\lambda_{2(N-1)}}{\lambda_{1N}} \leq 1$. Assume $\lambda_{2i}a_{i+1} \leq 1$, then $a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} \geq a_{i+1}$, and $a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} = (1 - \frac{\lambda_{2i}}{\lambda_{1i}})a_{i+1} + \frac{1}{\lambda_{1i}} \leq \frac{1}{\lambda_{1i}}$. Hence, using definition of a_i , we have $a_i = \frac{1}{\lambda_{1i}}$. Therefore, $\lambda_{2(i-1)}a_i = \frac{\lambda_{2(i-1)}}{\lambda_{1i}} \leq 1$. Theorem 3.3 implies that the greedy policy is optimal. \square

3.3 Lower Bound for Greedy Policy

In Section 3.2, we identified conditions under which the greedy policy is optimal. Unfortunately, as we saw in Section 3.1, the greedy policy is not always optimal. In these cases, it is important to identify the effectiveness of this policy. For this purpose, we derive a lower bound on the ratio of the value returned by greedy policy to the optimal value of (4)-(7).

Theorem 3.4. For any p , let $J_0(p)$ be the optimal value of (4)-(7) and $J_0^*(p)$ be the value returned by greedy policy. Denote $\mathcal{I} = \{i \in \{0, 1, \dots, N-1\} | \lambda_{2i} a_{i+1} \geq 1\}$, then

$$\frac{J_0^*(p)}{J_0(p)} \geq \prod_{i \in \mathcal{I}} \frac{1}{\lambda_{2i} a_{i+1}}. \quad (25)$$

Proof. For any $p < \bar{p}_i$, Let $q_i^* = \frac{\bar{p}_i - p}{\lambda_{1i}}$ and $\bar{q}_i = \operatorname{argmax}_{0 \leq q_i \leq \frac{\bar{p}_i - p}{\lambda_{1i}}} \{q_i + E(J_{i+1}(p + \lambda_{2i} q_i + \epsilon_i))\}$. Then for any i such that $\lambda_{2i} a_{i+1} \geq 1$, we have

$$\begin{aligned} \frac{q_i^* + E(J_{i+1}(p + \lambda_{2i} q_i^* + \epsilon_i))}{\bar{q}_i + E(J_{i+1}(p + \lambda_{2i} \bar{q}_i + \epsilon_i))} &\geq \frac{q_i^* + E(J_{i+1}(p + \lambda_{2i} q_i^* + \epsilon_i))}{\bar{q}_i + E(J_{i+1}(p + \lambda_{2i} q_i^* + \epsilon_i)) + a_{i+1} \lambda_{2i} (q_i^* - \bar{q}_i)} \\ &\geq \frac{q_i^* + E(J_{i+1}(p + \lambda_{2i} q_i^* + \epsilon_i))}{a_{i+1} \lambda_{2i} (q_i^* + E(J_{i+1}(p + \lambda_{2i} q_i^* + \epsilon_i)))} = \frac{1}{a_{i+1} \lambda_{2i}}, \end{aligned} \quad (26)$$

where the first inequality follows from (16) and second inequality follows from the assumption that $\lambda_{2i} a_{i+1} \geq 1$. Now we prove (25) by induction on the number of periods. Assume $\frac{J_1^*(p)}{J_1(p)} \geq \prod_{i \in \mathcal{I} \setminus \{0\}} \frac{1}{\lambda_{2i} a_{i+1}}$, then

$$\begin{aligned} J_0^*(p) = q_0^* + E(J_1^*(p + \lambda_{20} q_0^* + \epsilon_0)) &\geq q_0^* + \left(\prod_{i \in \mathcal{I} \setminus \{0\}} \frac{1}{\lambda_{2i} a_{i+1}} \right) E(J_1(p + \lambda_{20} q_0^* + \epsilon_0)) \\ &\geq \left(\prod_{i \in \mathcal{I} \setminus \{0\}} \frac{1}{\lambda_{2i} a_{i+1}} \right) (q_0^* + E(J_1(p + \lambda_{20} q_0^* + \epsilon_0))) \\ &\geq \left(\prod_{i \in \mathcal{I} \setminus \{0\}} \frac{1}{\lambda_{2i} a_{i+1}} \right) \frac{1}{\lambda_{20} a_1} (\bar{q}_0 + E(J_1(p + \lambda_{20} \bar{q}_0 + \epsilon_0))) \\ &= \left(\prod_{i \in \mathcal{I}} \frac{1}{\lambda_{2i} a_{i+1}} \right) J_0(p), \end{aligned} \quad (27)$$

where the second inequality follows from $\lambda_{2i} a_{i+1} \geq 1$ and third inequality follows from (26). \square

Observe that the theorem is noise-independent and price-independent, i.e., we do not make any assumptions on the noise ϵ_i , price p and how the trader sets the reservation prices \bar{p}_j in the later periods $j = 1, \dots, N$. As we can see, the performance of the greedy policy only depends on the temporary and permanent price impact parameters λ_{ki} , $k = 1, 2$ and $i = 0, 1, \dots, N-1$.

4 Conclusion and Future Research

In our model, the impact on the price is assumed to be a linear function of the order size. It will be interesting to see what results we can get from the nonlinear price impact models. In this paper, we let time be measured in discrete time. As all the discrete time models in the literature, the temporary price impact lasts only one period. An interesting future research is to consider continuous time model, and allow that the resilience or transient price impact can decay over longer interval.

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