Price and Capacity Competition

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Motivation: Communication and Transportation Networks

- Analysis of resource allocation in the presence of decentralized information, selfish users/administrative domains.
- Instead of a central control objective, model as a multi-agent decision problem: Game theory and economic market mechanisms.
- Recent interest: Quantification of efficiency loss, “Price of Anarchy, Stability” in “user games” as a guarantee on performance in decentralized and unregulated networked-systems.
- Question: Effects of prices/tolls and investment decisions on performance when they are set (partly) for profit maximization

This Paper

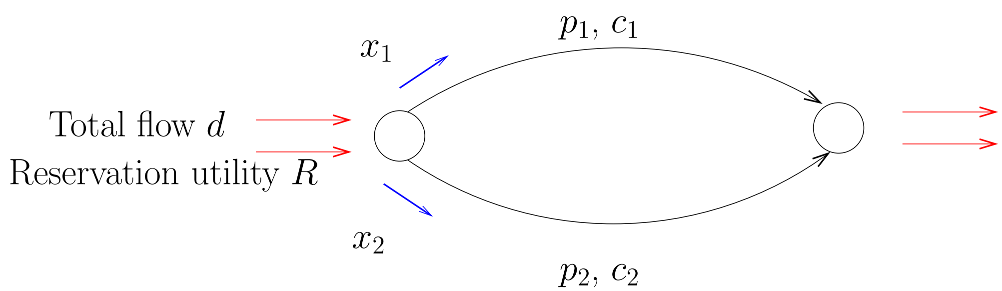
- A stylized model of price and capacity competition.
- Implications for timing of capacity and price choices for *existence of equilibrium* and *efficiency losses* in equilibrium.
- Three Main Sets of Results:
  1. **For a two-stage competition model**, where N firms invest in capacities first, and then compete in prices:
     - There exists (a continuum of) pure Oligopoly equilibria.
     - The Price of Anarchy (performance for the worst parameter values of the worst equilibrium) is 0.
     - The Price of Stability (performance for the worst parameter values of the best equilibria) bounded from below by $2\frac{\sqrt{N}-1}{N-1}$.
  2. A Stackelberg game for implementing the best equilibrium.
  3. **For a one-stage competition model**, where capacities and prices are chosen simultaneously:
     - A pure strategy Oligopoly equilibrium always fails to exist.

Related Work

- Industrial Organization Literature on Capacity Competition:
  - [Kreps and Scheinkman 83], [Davidson and Deneckere 84]: Main issue whether Bertrand competition yields Cournot outcomes and implications of the "rationing rule".
- Price Competition in Congested Networks:
  - [Acemoglu and Ozdaglar 05, 06], [Hayrapetyan, Tardos, and Wexler 05]: Bounds on the extent of inefficiency of unregulated price competition with congestion externalities.
- Investments and Price Competition:
  - [Weintraub, Johari, and Van Roy 06]: One-stage competition model in the presence of congestion externalities and in a symmetric environment.
- Today focus on capacity constraints without congestion externalities.

Model

- N firms competing over capacities and prices for user demand.
- Motivating Example: Service providers operating their own communication subnetworks.



- Interested in allocating d units of aggregate flow (of many "small users") between two firms.
- Users have a reservation utility R: they choose the lowest price firm until its capacity is reached; after this, remaining users allocate their capacity to second lowest price firm (as long as its price $\leq$ R); so on.
- Service provider i invests in capacity c_i at a cost of γ_i, and charges a price p_i per unit flow on link i.

Two-Stage Competition

- Model economy corresponds to the following 3-stage game:
  - First, N firms simultaneously choose their capacities.
  - Second, having observed the capacities, firms simultaneously choose their prices.
  - Finally, users allocate their demands across the firms.
- We refer to the dynamic game between the firms as price-capacity competition game.
- Good approximation to a situation in which prices can change at much higher frequencies than capacities.

User Demand

Definition: For a given capacity vector c and price vector $p \geq 0$, a vector $x^* \geq 0$ is a **flow equilibrium** if

$$x^* \in \arg \max_{\substack{0 \leq x_i \leq c_i \\ \sum_{i=1}^N x_i \leq 1}} \left\{ \sum_{i=1}^N (R - p_i)\, x_i \right\}. \qquad (1)$$

We denote the set of flow equilibrium at a given p and c by $W[p, c]$.

Proposition: Let $c = (c_1, ..., c_N)$ be a capacity vector and $p = (p_1, ..., p_N)$ be a price vector. Suppose that for some $M \leq N$, we have $p_1 < p_2 < ... < p_M \leq R < p_{M+1}$. Then, there exists a unique flow equilibrium $x \in W[p, c]$ given by

$$x_1 = \min\{c_1, 1\},$$

$$x_m = \min\left\{ c_m, \max\left\{0, 1 - \sum_{i=1}^{m-1} x_i \right\}\right\}, \qquad \forall\ 2 \leq m \leq M.$$

For 2 firms; when $p_1 < p_2 \leq R$, the unique flow equilibrium:

$$x_1 = \min\{c_1, 1\} \text{ and } x_2 = \min\{c_2, 1 - x_1\}.$$

Social Optimum

Definition: A capacity-flow vector (c^S, x^S) is a *social optimum* if it is an optimal solution of the *social problem*

$$\begin{aligned} \text{maximize}_{x\geq 0,\ c\geq 0} \quad & \sum_{i=1}^{N} R x_i - \sum_{i=1}^{N} \gamma_i c_i \qquad (2) \\ \text{subject to} \quad & \sum_{i=1}^{N} x_i \leq 1, \\ & x_i \leq c_i, \quad i \in \{1, \dots, N\}. \end{aligned}$$

The *social capacity* c^S is given as the solution to the following maximization problem:

$$c^S \in \arg \max_{c\geq 0,\ \sum c_i \leq 1} \left\{ \sum_{i=1}^{N} (R - \gamma_i) c_i \right\}. \qquad (3)$$

Price Equilibrium

- Given the price vector of other firms, p_{-i}, the profit of firm i is
$$\Pi_i[p_i, p_{-i}, c_i, c_{-i}, x] = p_i x_i - \gamma_i c_i,$$
where $x \in W[p, c]$ is a flow equilibrium given p and c.
- We look for the subgame perfect equilibria (SPE) of this game.

Definition: [Price Equilibrium] Given $c \geq 0$, a vector $[p(c), x(c)]$ is a *pure strategy Price Equilibrium* if $x(c) \in W[p(c), c]$ and for all i,

$$\Pi_i[p_i(c), p_{-i}(c), x(c), c] \geq \Pi_i[p_i, p_{-i}(c), x, c],\ \forall\, p_i \geq 0,\ x \in W[p_i, p_{-i}(c), c].$$

A vector $[\mu^c, x^c(p)]$ is a *mixed strategy Price Equilibrium* if $\mu^c \in \mathcal{B}^N$ and the function $x^c(p) \in W[p, c]$ for every p, and for all i and $\mu_i \in \mathcal{B}$,

$$\int_{[0,R]^N} \Pi_i[p_i, p_{-i}, x^c(p_i, p_{-i}), c]d\ \ \mu_i^c(p_i) \times \mu_{-i}^c(p_{-i})$$
$$\geq \int_{[0,R]^N} \Pi_i[p_i, p_{-i}, x^c(p_i, p_{-i}), c]d\ \ \mu_i(p_i) \times \mu_{-i}^c(p_{-i})\ \ .$$

We denote set of pure [mixed] price eq. at a given c by $PE(c)$ $[MPE(c)]$.

Oligopoly Equilibrium

- We next define the SPE of the entire game, focusing on the actions along the equilibrium path.
- We denote the profits of the mixed strategy price equilibria in the capacity subgame by

$$\Pi_i[\mu, x(\cdot), c] = \int_{[0,R]^N} \Pi_i[p, x(p), c] d\mu(p).$$

Definition: [Oligopoly Equilibrium] A vector $[c^{OE}, p(c^{OE}), x(c^{OE})]$ is a *(pure strategy) Oligopoly Equilibrium (OE)* if $[p(c^{OE}), x(c^{OE})] \in PE(c^{OE})$ and for all $i \in \{1, ..., N\}$,

$$\Pi_i[p(c^{OE}), x(c^{OE}), (c_i^{OE}, c_{-i}^{OE})] \geq \Pi_i[\mu, x(\cdot), (c_i, c_{-i}^{OE})], \tag{4}$$

for all $c_i \geq 0$, and for all $[\mu, x(\cdot)] \in MPE(c_i, c_{-i}^{OE})$. We refer to c^{OE} as the *OE capacity*.

Existence of Pure and Mixed Price Equilibria

We assume without loss of generality that $d = 1$ and $c_i > 0$ for all i.

- Suppose that $\sum_{i=1}^{N} c_i \leq 1$. Then there exists a unique PE in the capacity subgame $[p, x]$ such that $p_i = R$ and $x_i = c_i$.
- Suppose that $\sum_{i=1}^{N} c_i > 1$, and there exists some j with $\sum_{i=1}^{N} c_i - c_j < 1$. Then there exists no pure PE, but there exists a mixed strategy PE.
- Suppose that for each $j \in \{1, ..., N\}$, $\sum_{i=1}^{N} c_i - c_j \geq 1$. Then, for all PE $[p, x]$, we have $p_i = 0$ for $i \in \{1, ..., N\}$, i.e., all firms make (ex-post) zero profits.
  - Capacity subgame: uncapacitated Bertrand price competition.

Characterization of Mixed Price Equilibria

Denote the (essential) support of μ_i by $[l_i, u_i]$ and the corresponding cumulative distributions by F_i

Proposition: Let c be a capacity vector with $\sum_{i=1}^N c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and suppose that there exists j with $\sum_{i=1}^N c_i - c_j < 1$. Let $\bar{c} = \max_{i=1,\ldots,N} c_i$. Let $u = \max_{i\in\{1,2,\ldots,N\}} u_i$. For firm j, the expected profits $\Pi_j[\mu, x(\cdot), c]$ are given by

$$\Pi_j[\mu, x(\cdot), c] = \begin{cases} R(1 + \bar{c} - \sum_{i=1}^N c_i) - \gamma_j c_j, & \text{if } F_j \text{ has an atom at } u, \\ R(1 + \bar{c} - \sum_{i=1}^N c_i)\frac{c_j}{\bar{c}} - \gamma_j c_j, & \text{if } F_j \text{ has no atom at } u. \end{cases}$$

Example (Two firms)

Let $c = (c_1, c_2)$ be a capacity vector with $1 < c_1 + c_2 < 2$, and $c_i \leq 1$ for $i = 1, 2$. Let $[\mu, x(\cdot)]$ be a mixed PE in the capacity subgame c. The expected profits $\Pi_i[\mu, x(\cdot), c]$, for $i = 1, 2$ are given by

$$\Pi_i[\mu, x(\cdot), c] = \begin{cases} R(1 - c_{-i}) - \gamma_i c_i, & \text{if } c_{-i} \leq c_i, \\ \frac{R(1-c_i)c_i}{c_{-i}} - \gamma_i c_i, & \text{if } c_i \leq c_{-i}, \end{cases}.$$

Proof of the Proposition

Relies on two lemmas:

Lemma: Let l denote the minimum of the lower supports of the mixed strategies, i.e., $l = \min_{i \in \{1,2,\ldots,N\}} l_i$. Let P_l denote the set of firms whose lower support is l, i.e., $P_l = \{i \in \{1, \ldots, N\} \; : \; l_i = l\}$. Then:

(i) $\sum_{i \in P_l} c_i > 1$.

(ii) For all $i \in P_l$, F_i does not have an atom at l.

Lemma: Let u denote the maximum of the upper supports of the mixed strategies, i.e., $u = \max_{i \in \{1,2,\ldots,N\}} u_i$. Let $c_k \geq c_i$, for all $i \in \{1, \ldots, N\}$. Then:

(i) At most one distribution F_i can have an atom at u.

(ii) If the distribution F_i has an atom at u, then $c_i = c_k$.

(iii) The maximum upper support u is equal to R.

Any price in the support yields the same expected profits.

Existence and Characterization of OE

Proposition: Assume that $\gamma_i < R$ for some i. Let k be a firm with the maximum capacity, i.e., $c_k \geq c_i$ for all $k \in \{1, \ldots, N\}$. A capacity vector c is an OE capacity if and only if $\sum_{i=1}^{N} c_i = 1$ and

$$\frac{R - \gamma_i}{2R - \gamma_i} \cdot (c_i + c_k) \leq c_i \leq c_k, \tag{5}$$

for all $i \neq k$.

- This implies that there exists a continuum of OE capacities.
- For all $0 \leq \gamma_i \leq R$, the capacity vector $c = (1/N, \ldots, 1/N)$ satisfies the preceding.

Proposition: The price-capacity competition game has a pure strategy Oligopoly Equilibrium.

For two firms:

$$\frac{R - \gamma_i}{2R - \gamma_i} \leq c_i \leq c_{-i}, \quad \text{equivalently} \quad \frac{R - \gamma_1}{2R - \gamma_1} \leq c_1 \leq \frac{R}{2R - \gamma_2}.$$

Efficiency Analysis of OE

- Given capacity costs γ_i, let $C(\{\gamma_i\})$ denote the set of OE capacities. We define the efficiency metric at some $c^{OE} \in C(\{\gamma_i\})$ as

$$r(\{\gamma_i\}, c^{OE}) = \frac{\sum_{i=1}^{N} (R - \gamma_i) c_i^{OE}}{\sum_{i=1}^{N} (R - \gamma_i) c_i^{S}},$$

  where c^S is a social capacity given γ_i and the reservation utility R.

- We study:
  - The worst performance in a capacity equilibrium [Price of Anarchy (PoA)];

$$\inf_{\{0 \le \gamma_i \le R\}} \inf_{c^{OE} \in C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}).$$

  - The best performance in a capacity equilibrium [Price of Stability (PoS)],

$$\inf_{\{0 \le \gamma_i \le R\}} \sup_{c^{OE} \in C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}).$$

Efficiency Analysis of OE

The PoA of the price-capacity competition game is 0:

Example: For two firms, let $\gamma_1 = R - \epsilon$ for some $0 < \epsilon < \min\{1, R\}$, $\gamma_2 = R - \epsilon^2$:

$$c^S = (1, 0) \text{ with surplus } \mathbb{S}(c^S) = \epsilon.$$

$$c^{OE} = (\frac{\epsilon}{R+\epsilon}, \frac{R}{R+\epsilon}) \text{ with surplus } \mathbb{S}(c^{OE}) = \frac{\epsilon^2(1+R)}{R+\epsilon}.$$

Therefore, as $\epsilon \to 0$, the efficiency metric satisfies

$$\lim_{\epsilon \to 0} r(\{\gamma_i\}, c^{OE}) = \lim_{\epsilon \to 0} \frac{\epsilon(1+R)}{R+\epsilon} = 0.$$

Efficiency Analysis of OE

Theorem: Consider the price competition game with N firms, $N \geq 2$. Then, for all $0 \leq \gamma_i \leq R$, $i = 1, \ldots, N$, we have

$$\sup_{c^{OE} \in \ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}) \geq 2\frac{\sqrt{N} - 1}{N - 1}$$

i.e., the PoS of the price-capacity competition game is $2\frac{(\sqrt{N}-1)}{(N-1)}$ and this bound is tight.

Example: Let $\gamma_1 = \delta > 0$ and $\gamma_2 = (2 - \sqrt{2})R$:

$$c^S = (1, 0) \text{ with surplus } \mathbb{S}(c^S) = R.$$

It can be seen that the efficiency metric satisfies

$$\lim_{\delta \to 0} r(\{\gamma_i\}, c^{OE}) = 2\sqrt{2} - 2 \approx \frac{5}{6}.$$

Implementation : Stackelberg Leader Game

- To simplify the exposition, we focus on $N = 2$ firms.
- Consider a four-stage game, where the low-cost firm (say firm 1) acts as the Stackelberg leader and chooses the its capacity first.

Definition [Stackelberg Equilibrium]: For a given $c_1 \geq 0$, let $BR_2(c_1)$ denote the set of best response capacities for firm 2, i.e.,

$$BR_2(c_1) = \arg \max_{\substack{c_2 \geq 0 \\ [\mu, x(\cdot)] \in MPE(c_1, c_2)}} \Pi_2[\mu, x(\cdot), c_1, c_2].$$

A vector $[c^{SE}, p(c^{SE}), x(c^{SE})]$ is a *(pure strategy) Stackelberg Equilibrium (SE)* if $[p(c^{SE}), x(c^{SE})] \in PE(c^{SE})$, $c_2^{SE} \in BR_2(c_1^{SE})$, and

$$\Pi_1[p(c^{SE}), x(c^{SE}), c_1^{SE}, c_2^{SE}] \geq \Pi_1[\mu, x(\cdot), c_1, c_2],$$

for all $c_1 \geq 0$, $[\mu, x(\cdot)] \in MPE(c_1, c_2)$, and $c_2 \in BR_2(c_1)$.

Efficiency of Stackelberg Game

Theorem: Suppose that $\gamma_1 < \gamma_2 \le R$. Then there exists a unique pure strategy Stackelberg equilibrium.

Moreover, for all $0 \le \gamma_i \le R$, $i = 1, 2$, we have

$$\inf_{c^{SE} \in \, C(\{\gamma_i\})} r(\{\gamma_i\}, c^{SE}) = \sup_{c^{SE} \in \, C(\{\gamma_i\})} r(\{\gamma_i\}, c^{SE}) = 2\sqrt{2} - 2,$$

i.e., both the PoA and PoS of the Stackelberg game is $2\sqrt{2} - 2$ and this bound is tight.

Simultaneous Capacity-Price Selection Game

Definition: A vector $[c^*, p^*, x^*]$ is a *(pure strategy) one-stage Oligopoly Equilibrium (OE)* if $x^* \in W[p^*, c^*]$ and for all $i \in \{1, ..., N\}$,

$$\Pi_i[(p_i^*, p_{-i}^*), x^*, (c_i^*, c_{-i}^*)] \geq \Pi_i[(p_i, p_{-i}^*), x, (c_i, c_{-i}^*)],$$

for all $p_i \geq 0$, $c_i \geq 0$, and all $x \in W[(p_i, p_{-i}^*), (c_i, c_{-i}^*)]$.

Proposition: Consider N firms playing the one-stage game described above with $N \geq 2$. Given any γ_i, with $0 < \gamma_i < R$, $i \in \{1, ..., N\}$, there does not exist a one-stage Oligopoly Equilibrium.

Intuition:

- Assume $[c^*, p^*, x^*]$ is a one-stage OE, then $\sum_{i=1}^{N} c_i^* = 1$ and $p_i^* = R$.
- Consider the case $c_1^* = \epsilon$ for some $\epsilon > 0$ and the "double deviation" $(c_j, p_j) = (1, R - \delta)$ for some $\delta > 0$
  $[R - \delta - \gamma_j > (R - \gamma_j)(1 - \epsilon)]$.

Congestion in Networks

- Analysis so far focused only on capacity constraints.
- In addition to capacity constraints, a main concern in communication networks is congestion (source of delay and packet loss).
- Presence of congestion in particular routes or subnetworks complicates analysis of equilibria and efficiency both with and without capacity investments.
  - More data or traffic on a particular route exerts a negative externality on existing data or traffic (e.g. by increasing delay or probability of packet loss).

Price Competition with Congestion Externalities

- Outline of results from [Acemoglu, Ozdaglar 05, 06]:
- New Feature: A higher price results in traffic moving to an alternative route, but also increases congestion there, making it less attractive.
  - New source of markup in oligopolistic competition.
  - Greater competition may decrease efficiency.

Price Competition with Congestion Externalities - Continued

- Same model except that users utility is

$$\sum_{i=1}^{N}(R - l_i(x_i) - p_i)x_i,$$

  where $l_i(x_i)$ is a convex latency function measuring costs of delay and congestion on link i as a function of link flow x_i.

- Notion of pure and mixed Oligopoly equilibrium same as before.

- A flow vector x^S is a *social optimum* if

$$l_i(x_i^S) + x_i^S l_i'(x_i^S) \quad = \quad \min_{j \in \mathcal{I}}\{l_j(x_j^S) + x_j^S l_j'(x_j^S)\}, \qquad \forall\ i \text{ with } x_i^S > 0.$$

- $(l_i)'(x_i^S)x_i^S$: Marginal congestion cost, Pigovian tax.

Price Characterization with Parallel Links

- Oligopoly Prices: Let (p^{OE}, x^{OE}) be an OE. Then,

$$p_i^{OE} = (l_i)'(x_i^{OE})x_i^{OE} + \frac{\sum_{j \in \mathcal{I}_s} x_j^{OE}}{\sum_{j \notin \mathcal{I}_s} \frac{1}{l_j'(x_j^{OE})}}$$

- In particular, for two links, the OE prices are given by

$$p_i^{OE} = x_i^{OE}(l_1'(x_1^{OE}) + l_2'(x_2^{OE})).$$

  - Increase in price over the marginal congestion cost as a function of the latency of the other link.
  - Reflects the new source of market power because of the congestion externality.

Efficiency Bound for Parallel Links (without Capacity Investments)

- Efficiency metric: Given a set of latency functions $\{l_i\}$ and an equilibrium flow x^{OE}, we define the efficiency metric as

$$r(\{l_i\}, x^{OE}) = \frac{R\sum_{i=1}^{I} x_i^{OE} - \sum_{i=1}^{I} l_i(x_i^{OE})x_i^{OE}}{R\sum_{i=1}^{I} x_i^{S} - \sum_{i=1}^{I} l_i(x_i^{S})x_i^{S}}.$$

- Theorem: Consider a parallel link network. Then

$$r(\{l_i\}, x^{OE}) \geq \frac{5}{6}, \qquad \forall\ \{l_i\}_{i\in\mathcal{I}},\ x^{OE},$$

  and the bound is tight.

- Tight bound irrespective of the number of links and market structure.

Future Work

- Studied efficiency of equilibria where firms compete over capacities and prices.
- Importance of the sequence of decisions.
- Briefly discussed the effects of congestion externalities.
- Extensions to combine congestion costs with investment decisions.
  - Existence of (pure strategy) Oligopoly Equilibria with general latency functions.
  - Efficiency properties.