Flow Representations of Games: Near Potential Games and Dynamics

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Introduction

- Game-theoretic analysis has been used extensively in the study of networks for two major reasons:
  - Game-theoretic tools enable a flexible control paradigm where agents autonomously control their resource usage to optimize their own selfish objectives.
  - Even when selfish incentives are not present, game-theoretic models and tools provide potentially tractable decentralized algorithms for network control.
- Important reality check: Do game-theoretic models make approximately accurate predictions about behavior?

Game-Theoretic Predictions in the k-Beauty Game

- Consider the following game, often called the k- beauty game.
- Each of the n-players will pick an integer between 0 and 100.
- The person who is closest to k times the average of the group will win a prize, where $0 < k < 1$.
- The unique Nash equilibrium of this game is $(0, \ldots, 0)$ (in fact, this is the unique iteratively strict dominance solvable strategy profile).
- How do intelligent people actually play this game? (e.g. MIT students)
- First time play: Nobody is close to 0. When $k = 2/3$, winning bids are around 20-25.

Game-Theoretic Predictions in the k-Beauty Game

- Why? If you ask the students, they are "rational" in that they bid k times their expectation of the average, but they are not "accurate" in their assessment of what that average is.
- If the same group of people play this game a few more times, almost everybody bids zero; i.e., their expectations become accurate and they "learn" the Nash equilibrium.
- This is in fact the most common justification of Nash equilibrium predictions. But this type of convergence to a Nash equilibrium is not a general result in all games.
- In fact, examples of nonconvergence or convergence to non-Nash equilibrium play (in mixed strategies) easy to construct.

Potential Games

- Potential games are games that admit a "potential function" (as in physical systems) such that maximization with respect to subcomponents coincide with the maximization problem of each player.
- Nice features of potential games:
  - A pure strategy Nash equilibrium always exists.
  - Natural learning dynamics converge to a pure Nash equilibrium.
- Only a few games in economics, social sciences, or networks are potential games.

Motivation of Our Research

- Even if a game is not a potential game, it may be “close” to a potential game. If so, it may inherit some of the nice properties in an approximate sense.
- How do we determine whether a game is “close” to a potential game?
- What is the topology of the space of preferences?
- Are there “natural” decompositions of games?
- Can certain games be perturbed slightly to turn them into potential games?

Main Contributions

- Analysis of the global structure of preferences
  - Representation of finite games as flows on graphs
- Canonical decomposition: potential, harmonic, and nonstrategic components
- Projection schemes to find the components.
- Closed form solutions to the projection problem.
- Characterization of approximate equilibria of a game using equilibria of its potential component.
- Analysis of dynamics in a game using the convergence properties of the dynamics in its potential component
- Applications in a wireless power control problem.

Potential Games

- We consider finite games in strategic form, $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$.
- $\mathcal{G}$ is an exact potential game if there exists a function $\Phi : E \to \mathbb{R}$, where $E = \Pi_{m \in \mathcal{M}} E^m$, such that

$$\Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m}) = u^m(x^m, x^{-m}) - u^m(y^m, x^{-m}),$$

for all $m \in \mathcal{M}$, $x^m,\ y^m \in E^m$, and $x^{-m} \in E^{-m}$ ($E^{-m} = \Pi_{k \neq m} E^k$).
- Weaker notion: ordinal potential game, if the utility differences above agree only in sign.
- Potential Φ aggregates and explains incentives of all players.
- Examples: congestion games, etc.

Potential Games and Nash Equilibrium

- A strategy profile x is a Nash equilibrium if

$$u^m(x^m, x^{-m}) \geq u^m(q^m, x^{-m}) \qquad \text{for all } m \in \mathcal{M},\ q^m \in E^m.$$

- A global maximum of an ordinal potential game is a pure Nash equilibrium.
- Every finite potential game has a pure equilibrium.
- Many learning dynamics (such as better-reply dynamics, fictitious play, spatial adaptive play) "converge" to a pure Nash equilibrium in finite games. [Monderer and Shapley 96], [Young 98], [Marden, Arslan, Shamma 06, 07].
- When is a given game a potential game?
- More importantly, what are the obstructions, and what is the underlying structure?

Existence of Exact Potential

A path is a collection of strategy profiles $\gamma = (x_0, \ldots, x_N)$ such that x_i and x_{i+1} differ in the strategy of exactly one player where $x_i \in E$ for $i \in \{0, 1, \ldots N\}$. For any path γ, let

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(x_i) - u^{m_i}(x_{i-1}),$$

where m_i denotes the player changing its strategy in the ith step of the path. A path $\gamma = (x_0, \ldots, x_N)$ is closed if $x_0 = x_N$.

Theorem ([Monderer and Shapley 96])

A game $\mathcal{G}$ is an exact potential game if and only if for all closed paths, γ, $I(\gamma) = 0$. Moreover, it is sufficient to check closed paths of length 4.

Existence of Exact Potential

- Let $I(\gamma) \neq 0$, if potential existed then it would increase when the cycle is completed.
- The condition for existence of exact potential is linear. The set of exact potential games is a subspace of the space of games.
- The set of exact potential games is "small".

Theorem

Consider games with set of players $\mathcal{M}$, and joint strategy space $E = \prod_{m \in \mathcal{M}} E^m$.

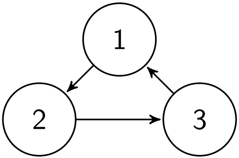
1. *The dimension of the space of games is $|\mathcal{M}| \prod_{m \in \mathcal{M}} |E^m|$.*
2. *The dimension of the subspace of exact potential games is*

$$\prod_{m \in \mathcal{M}} |E^m| + \sum_{m \in \mathcal{M}} \prod_{k \in \mathcal{M}, k \neq m} |E^k| - 1.$$

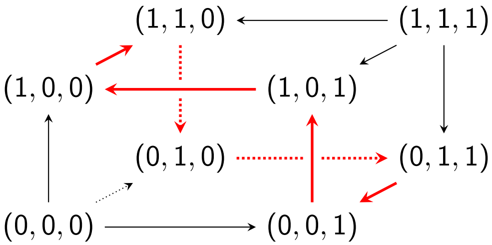
Existence of Ordinal Potential

- A weak improvement cycle is a cycle for which at each step, the utility of the player whose strategy is modified is nondecreasing (and at least at one step the change is strictly positive).
- A game is an ordinal potential game if and only if it contains no weak improvement cycles [Voorneveld and Norde 97].

Game Flows: 3-Player Example



- $E^m = \{0, 1\}$ for all $m \in \mathcal{M}$, and payoff of player i be -1 if its strategy is the same with its successor, 0 otherwise.
- This game is neither an exact nor an ordinal potential game.



Global Structure of Preferences

- What is the global structure of these cycles?
- Equivalently, topological structure of aggregated preferences.
- Conceptually similar to structure of (continuous) vector fields.
- A well-developed theory from algebraic topology, we need the combinatorial analogue for flows on graphs.

Decomposition of Flows on Graphs

- Consider an undirected graph $G = (E, A)$.
- We define the set of edge flows as functions $X : E \times E \to \mathbb{R}$ such that $X(\mathbf{p}, \mathbf{q}) = -X(\mathbf{q}, \mathbf{p})$ if $(\mathbf{p}, \mathbf{q}) \in A$, and 0 otherwise.
- Let C_0 denote the set of real-valued functions on the set of nodes, E, and C_1 denote the set of edge flows.
- We define the combinatorial gradient operator $\delta_0 : C_0 \to C_1$ as

$$(\delta_0 \phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q})(\phi(\mathbf{q}) - \phi(\mathbf{p})), \quad \mathbf{p}, \mathbf{q} \in E,$$

  where W is an indicator function for the edges of the graph, i.e., $W(x, y) = 1$ if $(x, y) \in A$, and 0 otherwise.
- We define the curl operator δ_1 as

$$(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \begin{cases} X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T, \\ 0 & \text{otherwise,} \end{cases}$$

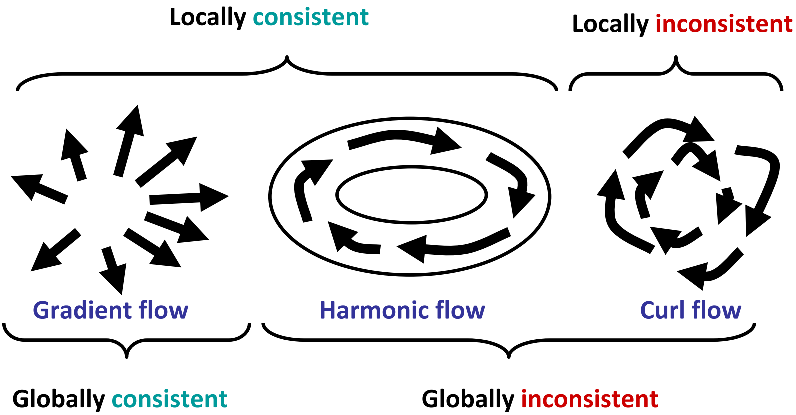
  where T is the set of *3-cliques* of the graph G (i.e., $T = \{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \mid (\mathbf{p}, \mathbf{q}), (\mathbf{q}, \mathbf{r}), (\mathbf{p}, \mathbf{r}) \in A\}$).

Helmholtz (Hodge) Decomposition

The Helmholtz Decomposition allows an orthogonal decomposition of the space of edge flows C_1 into three vector fields:

- Gradient flow: globally consistent component
  - An edge flow X is globally consistent if it is the gradient of some $f \in C_0$, i.e., $X = \delta_0 f$.
- Harmonic flow: locally consistent, but globally inconsistent component
  - An edge flow X is locally consistent if $(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}) = 0$ for all $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T$.
- Curl flow: locally inconsistent component

Helmholtz decomposition (a cartoon)



Flow Representations of Games

- A pair of strategy profiles that differ only in the strategy of player m are referred to as m-comparable strategy profiles.
- The set of comparable strategy profiles is the set of all such pairs (for all $m \in \mathcal{M}$).
- Notation:
  - The set of strategy profiles $E = \prod_{m \in \mathcal{M}} E^m$.
  - Set of pairs of m-comparable strategy profiles $A^m \subset E \times E$.
  - Set of pairs of comparable strategy profiles $A = \cup_m A^m \subset E \times E$.
- The game graph is defined as the undirected graph $G = (E, A)$, with set of nodes E and set of links A.

Flow Representations of Games – Continued

- For all $m \in \mathcal{M}$, let $W^m : E \times E \to \mathbb{R}$ satisfy

$$W^m(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{p}, \mathbf{q} \in A^m \\ 0 & \text{otherwise.} \end{cases}$$

- For all $m \in \mathcal{M}$, we define a difference operator D_m such that,

$$(D_m\phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q})\left(\phi(\mathbf{q}) - \phi(\mathbf{p})\right).$$

  where $\mathbf{p}, \mathbf{q} \in E$ and $\phi : E \to \mathbb{R}$.
- The flow generated by a game is given by $X = \sum_{m \in \mathcal{M}} D_m u^m$.

Strategically Equivalent Games

- Consider the following two games: Battle of the Sexes game and a slightly modified version.

	O	F
O	3, 2	0, 0
F	0, 0	2, 3

	O	F
O	4, 2	0, 0
F	1, 0	2, 3

- These games have the same "pairwise-comparisons", and therefore yield the same flows.
- To fix a representative for strategically equivalent games, we define the notion of games without any nonstrategic information.

Definition

We say that a game with utility functions $\{u^m\}_{m\in\mathcal{M}}$ does not contain any nonstrategic information if

$$\sum_{\mathbf{p}^m} u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = 0 \qquad \text{for all } \mathbf{p}^{-m} \in E^{-m},\ m \in \mathcal{M}.$$

Decomposition: Potential, Harmonic, and Nonstrategic

Decomposition of the game flows induces a similar partition of the space of games:

- When going from utilities to flows, the nonstrategic component is removed.
- Since we start from utilities (not preferences), always locally consistent.
- Therefore, two flow components: potential and harmonic

Thus, the space of games has a canonical direct sum decomposition:

$$G = G_{\mathrm{potential}} \oplus G_{\mathrm{harmonic}} \oplus G_{\mathrm{nonstrategic}},$$

where the components are orthogonal subspaces.

Bimatrix games

For two-player games, simple explicit formulas.
Assume the game is given by matrices (A, B), and (for simplicity), the non-strategic component is zero (i.e., $\mathbf{1}^T A = 0, B\mathbf{1} = 0$). Define

$$S := \frac{1}{2}(A + B), \quad D := \frac{1}{2}(A - B), \quad \Gamma := \frac{1}{2n}(A\mathbf{1}\mathbf{1}^T - \mathbf{1}\mathbf{1}^T B).$$

- Potential component:

$$(S + \Gamma, \quad S - \Gamma)$$

- Harmonic component:

$$(D - \Gamma, \quad -D + \Gamma)$$

Notice that the harmonic component is zero sum.

Projection on the Set of Exact Potential Games

- We solve,

$$d^2(\mathcal{G}) = \min_{\phi \in C_0} \quad ||\delta_0 \phi - \sum_{m \in \mathcal{M}} D_m u^m||_2^2,$$

  to find a potential function that best represents a given collection of utilities (recall C_0 is the space of real valued functions defined on E) .

- The utilities that represent the potential and that are close to initial utilities can be constructed by solving an additional optimization problem (for a fixed ϕ, and for all $m \in \mathcal{M}$):

$$\begin{aligned} \hat{u}^m = \arg\min_{\bar{u}^m} \quad & ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m \bar{u}^m = D_m \phi \\ & \bar{u}^m \in C_0. \end{aligned}$$

Projection on the Set of Exact Potential Games

Theorem

If all players have same number of strategies, the optimal projection is given in closed form by

$$\phi = \left(\sum_{m \in \mathcal{M}} \Pi_m \right)^{\dagger} \sum_{m \in \mathcal{M}} \Pi_m u^m,$$

and

$$\hat{u}^m = (I - \Pi_m) u^m + \Pi_m \left(\sum_{k \in \mathcal{M}} \Pi_k \right)^{\dagger} \sum_{k \in \mathcal{M}} \Pi_k u_k.$$

Here $\Pi_m = D_m^* D_m$ is the projection operator to the orthogonal complement of the kernel of D_m ($*$ denotes the adjoint of an operator).

Projection on the Set of Exact Potential Games

- The form of the potential function follows from the closed-form solution of a least-squares problem (i.e., the normal equation).
- For any $m \in \mathcal{M}$, $\Pi_m u^m$ and $(I - \Pi_m)u^m$ are respectively the strategic and nonstrategic components of the utility of player m.
- ϕ solves,

$$\sum_{m \in \mathcal{M}} \Pi_m \phi = \sum_{m \in \mathcal{M}} \Pi_m u^m.$$

  Hence, optimal ϕ is a function which represents the sum of strategic parts of utilities of different users.
- $\hat{u}^m$ is the sum of the nonstrategic part of u^m and the strategic part of the potential ϕ.

Wrapping Up

Nice canonical decomposition:

- Provides classes of games with simpler structures, for which stronger results can be proved.
- Yields a natural mechanism for approximation, for both static and dynamical properties.

Equilibria of a Game and its Projection

Theorem

Let $\mathcal{G}$ be a game and $\hat{\mathcal{G}}$ be its projection. Any equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ and any equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$ for $\epsilon \leq \sqrt{2} \cdot d(\mathcal{G})$.

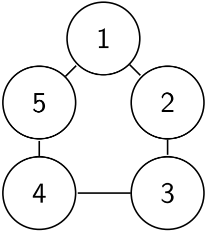
- Provided that the projection distance is small, equilibria of the projected game are close to the equilibria of the initial game.
- The projection framework can also be used to study convergence of dynamics in arbitrary games.
    - Will illustrate through a wireless power control application.
    - General result in the paper.

Simulation example

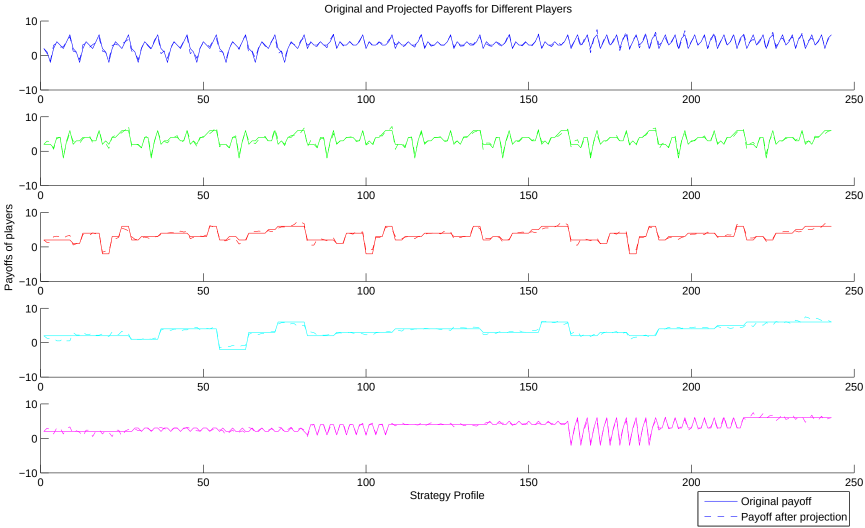
- Consider an average opinion game on a graph. Payoff of each player satisfies,

$$u^m(\mathbf{p}) = 2\hat{M} - (\hat{M}^m - \mathbf{p}^m)^2,$$

where $\hat{M}^m$ is the median of $\mathbf{p}^k$, $k \in N(m)$.



This game is not an exact (or ordinal) potential game.
With small perturbation in the payoffs, it can be projected to the set of potential games.


Original and Projected Payoffs for Different Players
Payoffs of players
Strategy Profile
Original payoff
Payoff after projection
10
0
−10
0
50
100
150
200
250

Wireless Power Control Application

- A set of mobiles (users) $\mathcal{M} = \{1, \ldots, M\}$ share the same wireless spectrum (single channel).
- We denote by $\mathbf{p} = (p_1, \ldots, p_M)$ the power allocation (vector) of the mobiles.
- Power constraints: $\mathcal{P}_m = \{p_m \mid \underline{P}_m \leq p_m \leq \bar{P}_m\}$, with $\underline{P}_m > 0$.
    - Upper bound represents a constraint on the maximum power usage
    - Lower bound represents a minimum QoS constraint for the mobile
- The rate (throughput) of user m is given by

$$r_m(\mathbf{p}) = \log\left(1 + \gamma \cdot \mathrm{SINR}_m(\mathbf{p})\right),$$

where, $\gamma > 0$ is the spreading gain of the CDMA system and

$$\mathrm{SINR}_m(\mathbf{p}) = \frac{h_{mm} p_m}{N_0 + \sum_{k \neq m} h_{km} p_k}.$$

Here, h_{km} is the channel gain between user k's transmitter and user m's receiver.

User Utilities and Equilibrium

- Each user is interested in maximizing a net rate-utility, which captures a tradeoff between the obtained rate and power cost:

$$u_m(\mathbf{p}) = r_m(\mathbf{p}) - \lambda_m p_m,$$

  where λ_m is a user-specific price per unit power.
- We refer to the induced game among the users as the power game and denote it by $\mathcal{G}$.
- Existence of a pure Nash equilibrium follows because the underlying game is a *concave game*.
- We are also interested in "approximate equilibria" of the power game, for which we use the concept of ϵ-(Nash) equilibria.
  - For a given ϵ, we denote by $\mathcal{I}_\epsilon$ the set of ϵ-equilibria of the power game $\mathcal{G}$, i.e.,

$$\mathcal{I}_\epsilon = \{\mathbf{p} \mid u_m(p_m, \mathbf{p}_{-m}) \geq u_m(q_m, \mathbf{p}_{-m}) - \epsilon, \quad \text{for all } m \in \mathcal{M},\ q_m \in \mathcal{P}_m\}$$

System Utility

- Assume that a central planner wishes to impose a general performance objective over the network formulated as

$$\max_{\mathbf{p} \in \mathcal{P}} U_0(\mathbf{p}),$$

  where $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$ is the joint feasible power set.
- We refer to $U_0(\cdot)$ as the system utility-function.
- We denote the optimal solution of this system optimization problem by p^* and refer to it as the desired operating point.
- Our goal is to set the prices such that the equilibrium of the power game can approximate the desired operating point p^*.

Potential Game Approximation

- We approximate the power game with a potential game.
- We consider a slightly modified game with player utility functions given by

$$\tilde{u}_m(\mathbf{p}) = \tilde{r}_m(\mathbf{p}) - \lambda_m p_m$$

  where $\tilde{r}_m(\mathbf{p}) = \log\left(\gamma \mathrm{SINR}_m(\mathbf{p})\right)$.
- We refer to this game as the potentialized game and denote it by $\tilde{\mathcal{G}} = \langle \mathcal{M}, \{\tilde{u}_m\}, \{\mathcal{P}_m\} \rangle$.
- For high-SINR regime (γ satisfies $\gamma \gg 1$ or $h_{mm} \gg h_{km}$ for all $k \neq m$),the modified rate formula $\tilde{r}_m(\mathbf{p}) \approx r_m(\mathbf{p})$ serves as a good approximation for the true rate, and thus $\tilde{u}_m(\mathbf{p}) \approx u_m(\mathbf{p})$.

Pricing in the Modified Game

Theorem

The modified game $\tilde{\mathcal{G}}$ is a potential game. The corresponding potential function is given by

$$\phi(\mathbf{p}) = \sum_m \log(p_m) - \lambda_m p_m.$$

- $\tilde{\mathcal{G}}$ has a unique equilibrium.
- The potential function suggests a simple linear pricing scheme.

Theorem

Let $\mathbf{p}^*$ be the desired operating point. Assume that the prices λ^* are given by

$$\lambda_m^* = \frac{1}{p_m^*}, \qquad \textit{for all } m \in \mathcal{M}.$$

Then the unique equilibrium of the potentialized game coincides with p^*.

Near-Optimal Dynamics

- We will study the dynamic properties of the power game $\mathcal{G}$ when the prices are set equal to λ^*.
- A natural class of dynamics is the best-response dynamics, in which each user updates his strategy to maximize its utility, given the strategies of other users.
- Let $\beta_m : \mathcal{P}_{-m} \to \mathcal{P}_m$ denote the best-response mapping of user m, i.e.,

$$\beta_m(\mathbf{p}_{-m}) = \arg \max_{p_m \in \mathcal{P}_m} u_m(p_m, \mathbf{p}_{-m}).$$

- Discrete time BR dynamics:

$$p_m \leftarrow p_m + \alpha \left(\beta_m(\mathbf{p}_{-m}) - p_m\right) \quad \text{for all } m \in \mathcal{M},$$

- Continuous time BR dynamics:

$$\dot{p}_m = \beta_m(\mathbf{p}_{-m}) - p_m \quad \text{for all } m \in \mathcal{M}.$$

- The continuous-time BR dynamics is similar to continuous time fictitious play dynamics and gradient-play dynamics [Flam, 2002], [Shamma and Arslan, 2005], [Fudenberg and Levine, 1998].

Convergence Analysis – 1

- If users use BR dynamics in the potentialized game $\tilde{\mathcal{G}}$, their strategies converge to the desired operating point p^*.
    - This can be shown through a Lyapunov analysis using the potential function of $\tilde{\mathcal{G}}$, [Hofbauer and Sandholm, 2000]
    - Our interest is in studying the convergence properties of BR dynamics when used in the power game $\mathcal{G}$.
- Idea: Use perturbation analysis from system theory
    - The difference between the utilities of the original and the potentialized game can be viewed as a perturbation.
    - Lyapunov function of the potentialized game can be used to characterize the set to which the BR dynamics for the original power game converges.

Convergence Analysis – 2

- Our first result shows BR dynamics applied to game $\mathcal{G}$ converges to the set of ϵ-equilibria of the potentialized game $\tilde{\mathcal{G}}$, denoted by $\tilde{\mathcal{I}}_\epsilon$.
- We define the minimum SINR:

$$\underline{SINR}_m = \frac{\underline{P}_m h_{mm}}{N_0 + \sum_{k \neq m} h_{km} \overline{P}_k}$$

- We say that the dynamics *converges uniformly* to a set S if there exists some $T \in (0, \infty)$ such that $\mathbf{p}^t \in S$ for every $t \geq T$ and any initial operating point $\mathbf{p}^0 \in \mathcal{P}$.

Lemma

The BR dynamics applied to the original power game $\mathcal{G}$ converges uniformly to the set $\tilde{\mathcal{I}}_\epsilon$, where ϵ satisfies

$$\epsilon \leq \frac{1}{\gamma} \sum_{m \in \mathcal{M}} \frac{1}{\underline{SINR}_m}.$$

- The error bound provides the explicit dependence on γ and $\underline{SINR}_m$.

Convergence Analysis – 3

- We next establish how "far" the power allocations in $\tilde{\mathcal{I}}_\epsilon$ can be from the desired operating point $\mathbf{p}^*$.

Theorem

For all ϵ, $\mathbf{p} \in \tilde{\mathcal{I}}_\epsilon$ satisfies

$$|\tilde{p}_m - p_m^*| \leq \overline{P}_m \sqrt{2\epsilon} \quad \textit{for every } \tilde{p} \in \tilde{\mathcal{I}}_\epsilon \textit{ and every } m \in \mathcal{M}$$

- Idea: Using the strict concavity and the additively separable structure of the potential function, we characterize $\tilde{\mathcal{I}}_\epsilon$.

Convergence and the System Utility

- Under some smoothness assumptions, the error bound enables us to characterize the performance loss in terms of system utility.

Theorem

Let $\epsilon > 0$ be given. (i) Assume that U_0 is a Lipschitz continuous function, with a Lipschitz constant given by L. Then

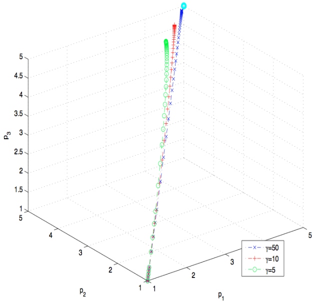
$$|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})| \leq \sqrt{2\epsilon} L \sqrt{\sum_{m \in \mathcal{M}} \overline{P}_m^2}, \quad \textit{for every } \tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon.$$

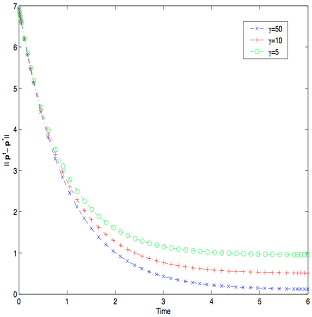
(ii) Assume that U_0 is a continuously differentiable function so that $|\frac{\partial U_0}{\partial p_m}| \leq L_m$, $m \in \mathcal{M}$. Then

$$|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})| \leq \sqrt{2\epsilon} \sum_{m \in \mathcal{M}} \overline{P}_m L_m, \quad \textit{for every } \tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon.$$

Numerical Example

- Consider a system with 3 users and let the desired operating point be given by $\mathbf{p}^* = [5, 5, 5]$.
- We choose the prices as $\lambda_m^* = \frac{1}{p_m^*}$ and pick the channel gain coefficients uniformly at random.
- We consider three different values of $\gamma \in \{5, 10, 50\}$.





Summary

- Analysis of the global structure of preferences
- Decomposition into potential and harmonic components
- Projection to “closest” potential game
- Preserves ϵ-approximate equilibria and dynamics
- Enables extension of many tools to non-potential games