

# Near-Optimal Power Control in Wireless Networks: A Potential Game Approach

Utku Ozan Candogan, Ishai Menache, Asuman Ozdaglar and Pablo A. Parrilo

Laboratory for Information and Decision Systems

Massachusetts Institute of Technology

Cambridge, MA, 02139

Emails: {candogan, ishai, asuman, parrilo}@mit.edu

**Abstract**—We study power control in a multi-cell CDMA wireless system whereby self-interested users share a common spectrum and interfere with each other. Our objective is to design a power control scheme that achieves a (near) optimal power allocation with respect to *any* predetermined network objective (such as the maximization of sum-rate, or some fairness criterion). To obtain this, we introduce the *potential-game approach* that relies on approximating the underlying noncooperative game with a “close” potential game, for which prices that induce an optimal power allocation can be derived. We use the proximity of the original game with the approximate game to establish through Lyapunov-based analysis that natural user-update schemes (applied to the original game) converge within a neighborhood of the desired operating point, thereby inducing near-optimal performance in a *dynamical* sense. Additionally, we demonstrate through simulations that the actual performance can in practice be very close to optimal, even when the approximation is inaccurate. As a concrete example, we focus on the sum-rate objective, and evaluate our approach both theoretically and empirically.

## I. INTRODUCTION

In contrast to wireline network architectures which can often provide quality of service (QoS) guarantees to end-users by strict division of the network resources, the shared nature of the wireless domain inherently implies that the performance of each mobile depends on the resources allocated to others. In code division multiple access (CDMA) systems, the transmission power of each mobile translates into interference noise for the other mobiles and thus degrades their performance in terms of the obtained data rates. Due to this mutual effect and in view of the scarcity of the power resource itself, power control has inarguably become a fundamental problem in wireless networks research. The power control problem, even when formulated as a centralized optimization problem with full information, is a fairly complex problem. For example, in general-topology CDMA network with multiple transmitters and receivers, each transmitter affects each of the receivers in a different manner, therefore it is not a priori clear how to assign the transmission powers in a system-efficient manner. Specifically, basic system objectives, such as the sum-rate, turn out to be difficult-to-solve optimization problems (see, e.g., [1]–[3] and references therein).

An additional concern within the power allocation framework is the possible *selfish* behavior of mobiles, who may autonomously control their transmission power to satisfy their

own interests. For example, each mobile may plausibly be interested in adjusting its power allocation in order to maximize its individual data-rate (throughput), while sustaining a physical-layer mandated power constraint. Naturally, game-theoretic tools have been widely applied over the last decade to study such competitive situations in wireless networks (see [4] and [5] for recent surveys). The agenda of bulk of the research in this area includes the study of the conditions for existence and uniqueness of a Nash equilibrium, and analysis of the stability properties of greedy power-updating schemes (see, e.g., [4], [6]–[8]).

In this paper, we consider the power allocation problem from the viewpoint of a central planner. The planner wishes to impose a certain power-dependent objective in the network, by properly pricing excessive power usage. As a concrete domain, we consider a multi-cell CDMA system with multiple transmitters (henceforth referred to as “mobiles” or “users”), each associated with a (possibly different) base-station. The mobiles, which share a common spectrum and interfere with each other, are interested in maximizing their net utility (throughput minus monetary costs) subject to individual power constraints.

Our objective is to provide a *general* distributed power control scheme that would achieve (or approximately achieve) *any* underlying system objective, despite the selfishness of the mobiles. We accomplish this using a novel *potential-game approach*, which entails approximating the original game with a potential game that has a (additively) separable structure in the individual power allocations. This enables design of a simple pricing scheme that induces the equilibrium of the potential game to coincide with the optimal power allocation of any underlying system objective. Moreover, since natural user-update schemes converge to a Nash equilibrium for potential games, the closeness of the two games allows us to establish near-optimal performance for user dynamics applied to the original game.

We apply the potential-game approach to the CDMA domain, by showing that the power game can be approximated by a potential game, with arbitrarily good accuracy as the signal to interference-plus-noise ratio (SINR) increases (e.g., when the interference due to other mobiles is negligible). We show that best-response dynamics applied to the original game converges within a neighborhood of the optimal operating point, where

the size of the neighborhood depends on the SINR. This shows that this approach can be used for network regulation under any SINR regime with explicit performance guarantees. We supplement the theory with experimental results, which demonstrate that the obtained performance can in practice be very close to optimal, even when operating at a relatively low-SINR regime.

*Related work.* There has been much work in the literature on pricing in communication networks in general, and wireless networks in particular. However, to the best of our knowledge, there is no general framework that tackles the problem of achieving (near) optimal performance for *any* given underlying system objective. One branch of this literature focuses on the profit maximization objective where service providers price usage of network resources to maximize their profits (see, e.g., [9]–[11]). Another branch assumes that prices are set by a social planner to regulate selfish user behavior and drive the system to a more desirable operating point (e.g., [7], [8], [12] and references therein).

*Paper structure.* The rest of the paper is organized as follows. The model is presented in Section II. We then proceed in Section III to present the potential-game approach. Section IV provides performance-loss bounds for the mobile interaction under a general SINR regime, and examines through simulations the actual performance of the pricing scheme. We also briefly describe in this section how to distribute the price generation process. As a concrete example, we focus in Section V on the sum-rate objective, and evaluate the resulting performance both theoretically and empirically. We conclude in Section VI.

## II. THE MODEL

### A. Preliminaries

We consider a set of mobiles  $\mathcal{M} = \{1, \dots, M\}$  that share the same wireless spectrum. Each mobile  $m \in \mathcal{M}$  transmits to a pre-assigned base station (we allow for multiple mobiles transmitting to the same base station).

Denote by  $\mathbf{p} = (p_1, \dots, p_M)$  the *power allocation* of the mobiles (we shall also refer to  $\mathbf{p}$  as the *operating point* of the network). The basic performance measure that determines the mobiles' rates is their SINR. The SINR of user  $m$  is given by

$$\text{SINR}_m(\mathbf{p}) = \frac{h_{mm}p_m}{N_0 + \sum_{k \neq m} h_{km}p_k}, \quad (1)$$

where  $h_{km}$  is the gain between user  $k$ 's transmitter and user  $m$ 's base station, and  $N_0$  is the noise power. Note that in CDMA systems, the signals of other users are often treated as interfering noise (i.e., there is no interference cancellation at the receiver), hence the user rate directly depends on its SINR, as we elaborate below.

Over the time-period of interest, we assume that the channel gains are fixed (i.e., fading effects take place at a much slower time-scale). Let  $r_m(\mathbf{p})$  be the rate (throughput) of user  $m$ , as determined by the power allocation  $\mathbf{p} = (p_1, \dots, p_M)$ . Then,

$$r_m(\mathbf{p}) = \log(1 + \gamma \text{SINR}_m(\mathbf{p})), \quad (2)$$

where  $\gamma > 0$  is the spreading gain of the CDMA system. The rate function  $r_m(\cdot)$  can be interpreted as being proportional to the Shannon capacity of user  $m$ , while we make the simplifying assumption that the noise plus the interference of all other users constitute an independent Gaussian noise.

It is assumed that each user  $m$  can adjust its transmission power  $p_m$  within a bounded range

$$0 < \underline{P}_m \leq p_m \leq \bar{P}_m. \quad (3)$$

The upper bound  $\bar{P}_m$  represents a constraint on the maximal usage of the battery, while the lower bound  $\underline{P}_m > 0$  is the lowest power level at which the user's transmitter can operate. In practice, a transmitter would have a finite number of power levels in a bounded range, however we assume for simplicity that any power in the above range can be sustained<sup>1</sup>.

### B. Utilities and Equilibrium

A basic modelling assumption in our work is that users are selfish and adjust their power in a self interested manner. As a result, the individual power usage has to be regulated in some way. To that end, we consider in this paper the following user-based linear pricing scheme: User  $m$  pays  $c_m$  monetary units per-power unit. The prices  $c_m$  are set in accordance with a global network objective (see Section II-C).

Note that the overall payment of user  $m$  is given by  $c_m p_m$ . The user objective is to maximize a net rate-utility, which captures a tradeoff between the obtained rate and the monetary cost, given by

$$u_m(\mathbf{p}) = r_m(\mathbf{p}) - \zeta_m c_m p_m, \quad (4)$$

where  $\zeta_m > 0$  is a user-specific rate vs. money trade-off coefficient. The individual user optimization problem, fixing other users' power allocation  $\mathbf{p}_{-m} \triangleq (p_1, \dots, p_{m-1}, p_{m+1}, \dots, p_M)$ , can be formulated as

$$\max_{\tilde{p}_m \in \mathcal{P}_m} u_m(\tilde{p}_m, \mathbf{p}_{-m}), \quad (5)$$

where  $\mathcal{P}_m = \{p_m \mid \underline{P}_m \leq p_m \leq \bar{P}_m\}$ . We refer to  $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_M$  as the joint feasible strategy space, and to  $\mathcal{P}_{-m} = \mathcal{P}_1 \times \dots \times \mathcal{P}_{m-1} \times \mathcal{P}_{m+1} \times \dots \times \mathcal{P}_M$  as the joint feasible strategy space of all users but the  $m$ -th one.

We formally denote a game instance by  $\mathcal{G} = \langle \mathcal{M}, \{u_m\}_{m \in \mathcal{M}}, \{\mathcal{P}_m\}_{m \in \mathcal{M}} \rangle$  and refer to this game as the *power game*. A *Nash equilibrium (NE)* of  $\mathcal{G}$  is a feasible power allocation  $\mathbf{p} \in \mathcal{P}$  from which no user has an incentive to unilaterally deviate, namely

$$u_m(\mathbf{p}) \geq u_m(\tilde{p}_m, \mathbf{p}_{-m}) \quad (6)$$

for every  $\tilde{p}_m \in \mathcal{P}_m$ . The existence of a Nash equilibrium follows in view of the fact that the underlying game is a *concave game* [13] (i.e.,  $u_m(\cdot)$  is concave in  $p_m$  and the joint strategy space  $\mathcal{P}$  is convex).

<sup>1</sup>In general, a user could decide not to use its power at all. We exclude this option here, by implicitly assuming that each user must sustain a nonzero lower bound on its rate, which is guaranteed to be satisfied by transmitting at a power of at least  $\underline{P}_m > 0$ .

In this paper we also consider operating points which are approximately Nash equilibria. To formally address such operating points, we use the concept of  $\epsilon$ -(Nash) equilibrium. An operating point  $\mathbf{p}$  is an  $\epsilon$ -equilibrium of the game  $\mathcal{G}$  if for every  $q_m \in \mathcal{P}_m$  and  $m \in \mathcal{M}$

$$u_m(p_m, \mathbf{p}_{-m}) \geq u_m(q_m, \mathbf{p}_{-m}) - \epsilon. \quad (7)$$

### C. System Utility

Assume that a central planner wishes to impose some performance objective over the network. Generally, the objective relates to the transmission powers employed by the users, hence can alternatively be posed as an optimization problem

$$\max_{\mathbf{p} \in \mathcal{P}} U_0(\mathbf{p}), \quad (8)$$

where  $U_0(\cdot)$  is the *system utility-function*. As a concrete example, we will consider in Section V the sum-rate objective, given by  $U_0(\mathbf{p}) = \sum_m r_m(\mathbf{p})$ , where  $r_m(\cdot)$  is defined in (2).

For the analysis in this paper, we shall assume that the central planner is equipped with the required information to solve (8) (i.e., the knowledge of all channel gains and the feasible power region), and actually is able to solve this optimization problem. We denote the optimal solution of (8) by  $\mathbf{p}^*$ , and refer to it henceforth as the *desired operating point*. In the sequel, we consider the price setting problem faced by the central planner, who is interested in inducing the optimal utility value  $U_0(\mathbf{p}^*)$ .

## III. THE POTENTIAL-GAME APPROXIMATION

As an intermediate step in our analysis, we consider in this section a noncooperative game with *modified utilities*

$$\tilde{u}_m(\mathbf{p}) = \tilde{r}_m(\mathbf{p}) - \zeta_m c_m p_m, \quad (9)$$

where

$$\tilde{r}_m(\mathbf{p}) = \log(\gamma \text{SINR}_m(\mathbf{p})). \quad (10)$$

We refer to this game as the *potentialized game* and denote it by  $\tilde{\mathcal{G}} = \langle \mathcal{M}, \{\tilde{u}_m\}_{m \in \mathcal{M}}, \{\mathcal{P}_m\}_{m \in \mathcal{M}} \rangle$ .

When the spreading gain  $\gamma$  satisfies  $\gamma \gg 1$  (or alternatively  $h_{mm} \gg h_{km}$  for all  $k \neq m$ ), we say that users operate in high-SINR regime. Note that under this regime, the modified rate formula  $\tilde{r}_m(\mathbf{p}) \approx r_m(\mathbf{p})$  serves as a good approximation for the true rate, and thus  $\tilde{u}_m(\mathbf{p}) \approx u_m(\mathbf{p})$ .

### A. Properties of the Potentialized Game

In this subsection, we obtain some basic properties of the potentialized game  $\tilde{\mathcal{G}}$ . Specifically, we show that a Nash equilibrium point for the game always exists and is *unique*. Furthermore, we establish that the game  $\tilde{\mathcal{G}}$  is a potential game.

A Nash equilibrium for the potentialized game is defined as in (6), with  $u_m(\mathbf{p})$  replaced by  $\tilde{u}_m(\mathbf{p})$ . Noting that  $\tilde{u}_m(\mathbf{p})$  remains (strictly) concave in  $p_m$  and that the feasible strategy space remains  $\mathcal{P}$ , the existence of a NE is guaranteed [13]. We proceed to show that  $\tilde{\mathcal{G}}$  belongs to the class of *potential games*. A game  $\tilde{\mathcal{G}} = \langle \mathcal{M}, \{\tilde{u}_m\}_{m \in \mathcal{M}}, \{\mathcal{P}_m\}_{m \in \mathcal{M}} \rangle$  is said to

be a potential game if there exists a function  $\phi : \mathcal{P} \rightarrow \mathbb{R}$  satisfying

$$\phi(p_m, \mathbf{p}_{-m}) - \phi(q_m, \mathbf{p}_{-m}) = \tilde{u}_m(p_m, \mathbf{p}_{-m}) - \tilde{u}_m(q_m, \mathbf{p}_{-m}), \quad (11)$$

for every  $m \in \mathcal{M}$ ,  $p_m, q_m \in \mathcal{P}_m$ ,  $\mathbf{p}_{-m} \in \mathcal{P}_{-m}$ .

*Proposition 1:* The game  $\tilde{\mathcal{G}}$  is a potential game. The corresponding potential function is given by

$$\phi(\mathbf{p}) = \sum_m \log(p_m) - \zeta_m c_m p_m. \quad (12)$$

*Proof:* This follows using the characterization of potential games in [14], i.e.,  $\frac{\partial \phi(\mathbf{p})}{\partial p_m} = \frac{\partial \tilde{u}_m(\mathbf{p})}{\partial p_m}$ ,  $m \in \mathcal{M}$ . ■

The uniqueness of the equilibrium now follows by exploiting the potential formula (12).

*Proposition 2:* The potentialized game  $\tilde{\mathcal{G}}$  has a unique Nash equilibrium.

*Proof:* Note that the potential function (12) is strictly concave and continuously differentiable. It is shown in [15] that if the potential function is concave and continuously differentiable and if the users' strategy spaces are convex (intervals in our case), then the set of pure strategy Nash equilibria coincides with the set of maximizers of the potential function. The potential function (12) is strictly concave over a convex strategy-space, hence the maximizer is unique, implying that the NE is unique. ■

### B. Assigning Prices

Our interest in this subsection is in deriving prices  $\mathbf{c}^* = (c_1^*, \dots, c_M^*)$  for the potentialized game  $\tilde{\mathcal{G}}$  such that the unique equilibrium of  $\tilde{\mathcal{G}}$  will coincide with the desired operating point  $\mathbf{p}^*$  (recall that the prices affect the utilities of the game). As mentioned earlier, we do not consider here how  $\mathbf{p}^*$  is obtained and assume it has been calculated by a central planner; we show below that equipped with  $\mathbf{p}^* \in \mathcal{P}$ , the central planner can set the prices  $\mathbf{c}^*$  in a simple way.

*Theorem 3:* Let  $\mathbf{p}^*$  be the desired operating point. Then the prices  $\mathbf{c}^*$  are given by

$$c_m^* = (\zeta_m p_m^*)^{-1}, \quad m \in \mathcal{M}. \quad (13)$$

*Proof:* We show that when  $c_m = c_m^*$  for every user  $m$ , then  $\mathbf{p}^*$  is the unique equilibrium of the potentialized game. Since  $\tilde{\mathcal{G}}$  is a potential game, the maximum of its potential  $\phi$  is a Nash equilibrium. We next show that  $\mathbf{p}^*$  is a maximum of the potential. Using  $\{c_m^*\}$  as the prices, the partial derivative of the potential is given by

$$\frac{\partial \phi(\mathbf{p})}{\partial p_m} = \frac{1}{p_m} - \zeta_m \frac{1}{\zeta_m p_m^*} = \frac{1}{p_m} - \frac{1}{p_m^*}.$$

Setting  $\mathbf{p} = \mathbf{p}^*$ , we thus have  $\frac{\partial \phi(\mathbf{p}^*)}{\partial p_m} = 0$  for all  $m \in \mathcal{M}$ . Recalling that  $\phi$  is concave, it follows that  $\mathbf{p}^*$  is a global maximum of the potential; hence  $\mathbf{p}^* \in \mathcal{P}$  is an equilibrium of  $\tilde{\mathcal{G}}$ , which is unique by Proposition 2. ■

Adopting the price vector  $\mathbf{c}^*$  (given in (13)), the desired operating point  $\mathbf{p}^*$  is generally not an equilibrium of the power game  $\mathcal{G}$ . Nonetheless, due to the relation between the

games  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ , employing  $\mathbf{c}^*$  as the per-user prices induces *near-optimal* performance in  $\mathcal{G}$ , in the sense that natural game dynamics converges within a neighborhood of  $\mathbf{p}^*$ . This property is formalized in the next section.

#### IV. NEAR-OPTIMAL DYNAMICS

In this section, we analyze the dynamical properties of the game  $\mathcal{G}$ , for which the per-user prices are set according to (13). Our main results herein are Theorems 4–5, which establish that best-response dynamics for  $\mathcal{G}$  converges within a neighborhood of the desired operating point  $\mathbf{p}^*$ , and consequently induce near-optimal performance in terms of the system utility  $U_0$ . The section is organized as follows. We describe best-response dynamics in Section IV-A and analyze their properties in Section IV-B. We then provide in Section IV-C some numerical examples to highlight several aspects of the dynamics. We conclude this section by a brief discussion of the results and some practical aspects of our method. The technical proofs for this section can be found in the Appendix.

##### A. Best-Response Dynamics

A natural class of dynamics in multiuser noncooperative systems is the so-called *best-response* dynamics, in which each player updates its strategy to maximize its utility, given the strategies of other players. In our specific context, let  $\beta_m : \mathcal{P}_{-m} \rightarrow \mathcal{P}_m$  denote the best response mapping for the  $m$ th user, which satisfies

$$\beta_m(\mathbf{p}_{-m}) = \operatorname{argmax}_{p_m \in \mathcal{P}_m} u_m(p_m, \mathbf{p}_{-m}). \quad (14)$$

Note that best response maps are in general set-valued, however in our setting  $\beta_m(\mathbf{p}_{-m})$  is single-valued due to strict concavity of each user's utility in its strategy.

We assume that users update their power allocation in accordance with their best-response. Specifically, we assume the following update rule:

$$p_m \leftarrow p_m + \alpha (\beta_m(\mathbf{p}_{-m}) - p_m) \quad \text{for all } m \in \mathcal{M},$$

where  $\alpha > 0$  is a fixed step-size. Assuming that users update their power allocation frequently enough and for small  $\alpha$ , the above update rule may be approximated by the differential equation

$$\dot{p}_m = \beta_m(\mathbf{p}_{-m}) - p_m \quad \text{for all } m \in \mathcal{M}. \quad (15)$$

This continuous-time dynamics is similar to continuous time fictitious play dynamics and gradient-play dynamics (see, e.g., [16] and [17]). We henceforth refer to (15) as the *best-response (BR) dynamics* of our game. We note that if the users were to play the potentialized game, this dynamics would converge to  $\mathbf{p}^*$ . This observation can be easily shown through a Lyapunov analysis using the potential function of  $\tilde{\mathcal{G}}$  (e.g., by adapting the analysis in [18] to BR dynamics). Yet, our interest is in studying the dynamical properties of the power game  $\mathcal{G}$ , which is the subject of the next subsection.

##### B. Convergence Analysis

We study in this subsection the properties of best-response dynamics (15). Before proceeding with the analysis, we require additional notations and definitions.

Since our solution method compares the outcomes of the power game  $\mathcal{G}$  to those of the potentialized game  $\tilde{\mathcal{G}}$ , we need to define the user's best-response in the latter. Because  $\tilde{\mathcal{G}}$  is a potential game, it follows by definition (see (11)) that the corresponding best-response  $\tilde{\beta}_m(\mathbf{p}_{-m})$  of any user can be obtained by maximizing the potential function  $\phi$  given the other users' strategies. Thus,

$$\tilde{\beta}_m(\mathbf{p}_{-m}) = \operatorname{argmax}_{p_m \in \mathcal{P}_m} \phi(p_m, \mathbf{p}_{-m}). \quad (16)$$

Note that  $\tilde{\beta}_m(\mathbf{p}_{-m})$  is also a single-valued function (by the strict concavity of the potential function).

We show below that the BR dynamics (15) operates in a neighborhood of  $\mathbf{p}^*$ . To formalize this property, we introduce the notion of uniform convergence (see [19] for related concepts). Let  $\mathbf{p}^t$  be the operating point at time  $t$ . We say that the dynamics *converges uniformly* to a set  $S$  if there exists some  $T \in (0, \infty)$  such that for any initial operating point  $\mathbf{p}^0 \in \mathcal{P}$ ,  $\mathbf{p}^t \in S$  for every  $t \geq T$ . In our context, the sets of interest relate to the equilibrium of the potentialized game. Specifically, for any given  $\epsilon$ , denote by  $\tilde{\mathcal{I}}_\epsilon$  the set of  $\epsilon$ -equilibria of  $\tilde{\mathcal{G}}$ , namely

$$\tilde{\mathcal{I}}_\epsilon = \{ \mathbf{p} \mid \tilde{u}_m(p_m, \mathbf{p}_{-m}) \geq \tilde{u}_m(q_m, \mathbf{p}_{-m}) - \epsilon \quad \text{for every } q_m \in \mathcal{P}_m \text{ and } m \in \mathcal{M} \}. \quad (17)$$

Our first result establishes that the dynamics (15) converges uniformly to a set  $\tilde{\mathcal{I}}_\epsilon$ , where  $\epsilon$  is explicitly characterized by the game parameters. Let

$$\underline{\text{SINR}}_m = \frac{P_m h_{mm}}{N_0 + \sum_{k \neq m} h_{km} \bar{P}_k}$$

be the minimal SINR of user  $m$ . Then,

*Lemma 1:* The best-response dynamics (15) in  $\mathcal{G}$  converges uniformly to  $\tilde{\mathcal{I}}_\epsilon$  (i.e., the set of  $\epsilon$ -equilibria of  $\tilde{\mathcal{G}}$ ), where  $\epsilon$  satisfies

$$\epsilon \leq \frac{1}{\gamma} \sum_{m \in \mathcal{M}} \frac{1}{\underline{\text{SINR}}_m}. \quad (18)$$

The proof of the lemma follows from a Lyapunov-based analysis, where the Lyapunov function used is related to the potential function  $\phi$  of the potentialized game  $\tilde{\mathcal{G}}$  (see Appendix for the proof)<sup>2</sup>. Observe that  $\epsilon$  above is inversely proportional to both  $\gamma$  and  $\underline{\text{SINR}}_m$ . This form of dependence is expected from the rate formula (2), as for large values of these terms we have  $\tilde{u}_m(\mathbf{p}) \approx u_m(\mathbf{p})$ . We note that the above lemma still does not provide an answer to how “far” the set of  $\epsilon$ -equilibria of  $\tilde{\mathcal{G}}$  is from the desired operating point  $\mathbf{p}^*$ . However, it is used below to establish that any point in the set  $\tilde{\mathcal{I}}_\epsilon$  (to which the BR dynamics converges) is in a neighborhood of  $\mathbf{p}^*$ .

<sup>2</sup>It also follows that the BR dynamics converges to a set of  $\epsilon$ -equilibrium of the power game  $\mathcal{G}$ . This result is omitted here due to lack of space, and can be found in the accompanying technical report for this paper [20].

*Theorem 4:* Let  $\tilde{\mathcal{I}}_\epsilon$  be given by (17), where  $\epsilon$  satisfies (18). Then  $|\tilde{p}_m - p_m^*| \leq \bar{P}_m \sqrt{2\epsilon}$  for every  $\tilde{p} \in \tilde{\mathcal{I}}_\epsilon$  and every  $m \in \mathcal{M}$ .

Under smoothness assumptions on the system utility  $U_0$ , a small  $\epsilon$  leads to near-optimal performance in terms of system utility. This is stated in the next theorem (the proof of which immediately follows from Theorem 4, see [20] for details).

*Theorem 5:* Let  $\tilde{\mathcal{I}}_\epsilon$  be given by (17), where  $\epsilon$  satisfies (18). (i) Assume that  $U_0$  is a Lipschitz continuous function, with a Lipschitz constant given by  $L$ . Then

$$|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})| \leq \sqrt{2\epsilon} L \sqrt{\sum_{m \in \mathcal{M}} \bar{P}_m^2} \quad (19)$$

for every  $\tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon$ .

(ii) Assume that  $U_0$  is a continuously differentiable function such that  $|\frac{\partial U_0}{\partial p_m}| \leq L_m$ ,  $m \in \mathcal{M}$ . Then

$$|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})| \leq \sqrt{2\epsilon} \sum_{m \in \mathcal{M}} \bar{P}_m L_m, \quad (20)$$

for every  $\tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon$ .

The expression  $|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})|$  can be regarded as a performance-loss measure. Theorem 5 implies that the bound on the performance-loss decreases with  $\epsilon$ . This is expected, since by Theorem 4, a small value of  $\epsilon$  implies that BR dynamics converges to a small neighborhood of  $\mathbf{p}^*$ ; hence, the dynamics operates in the proximity of the desired operating point. On the other hand, the bound increases with  $L$  (or  $L_m$ ), as the difference between  $\mathbf{p}^*$  and  $\tilde{\mathbf{p}}$  is translated to a difference in the associated values of  $U_0$  (which is scaled proportionally to  $L$  or  $L_m$ ).

We emphasize that Theorems 4–5 hold without requiring any assumptions on the SINRs of the users. Consequently, the performance bounds we obtain are valid for *any* choice of the system parameters  $\{h_{km}\}$ ,  $\{\underline{P}_m\}$ ,  $\{\bar{P}_m\}$ ,  $\gamma$ .

### C. Numerical Examples

Our objective in this subsection is to validate the actual performance of the suggested pricing scheme through basic experiments. Specifically, we are interested in examining how close we get in practice to the desired operating point, as a function of the “accuracy” of the potential-game approximation. As discussed in Section III (and also verified through the theoretical bounds of the preceding section), the approximation becomes better for larger spreading gain  $\gamma$ , or alternatively when the self-gain coefficients  $h_{mm}$  are much larger than the cross-gain coefficients  $h_{km}$  and the noise power  $N_0$ . For simplicity of implementation, we execute the simulations below for different values of  $\gamma$ , rather than significantly modifying the ratio between the self-gain coefficients and the cross-gain coefficients<sup>3</sup>.

<sup>3</sup>Typical values of the spreading gain in CDMA systems may range between 5–300 [7], [21]. In our experiments, we focus on a lower subset of this range, as when the spreading gain is above 100, the actual performance is indistinguishable from the optimal one.

We now describe the setup used for the experiments. We consider a network with three users. For all simulations, we assume that the desired operating point is  $\mathbf{p}^* = [5, 5, 5]$  and that the prices are set as in Theorem 3. Note that we actually do not specify here the underlying system utility, as our main concern in this set of simulations is to observe how close to  $\mathbf{p}^*$  best-response dynamics eventually converges. Since the gain coefficients and  $N_0$  can be scaled without changing the SINR, we normalize  $N_0$  to 1. The self-gain coefficients  $h_{mm}$  are chosen uniformly at random (from the interval  $[2, 4]$ ), and so do the cross-gain coefficients  $h_{km}$  (from the interval  $[0, 2]$ ). We consider three different values of  $\gamma$ ,  $\{5, 10, 50\}$ . We assume that  $\underline{P}_m = 1$ ,  $\bar{P}_m = 10$  for each player  $m \in \mathcal{M}$ . The dynamics are initialized at the operating point  $\mathbf{p}^0 = [1, 1, 1]$ .

We next examine the evolution of the operating point in time from two different angles. Figure 1 shows the time evolution of the operating point for different values of  $\gamma$ , starting from  $\mathbf{p}^0$  and ending in a neighborhood of  $\mathbf{p}^*$  (depending on the value of  $\gamma$ ). As expected, the dynamics tends to converge closer to  $\mathbf{p}^*$  as  $\gamma$  increases. Figure 2 depicts the  $L_2$ -norm distance of  $\mathbf{p}^t$  from  $\mathbf{p}^*$ . We observe that larger  $\gamma$ 's lead to shorter distances from  $\mathbf{p}^*$ , not only as a final outcome, but also at any point in time. Note further that all curves are monotonously decreasing, i.e., the dynamics tend to get consistently closer to the desired operating point, which is a desired property. An additional observation which is worth noting is that the trajectories of the dynamics seem to converge to a point (rather than to a larger set). This was not guaranteed by our theoretical results. An interesting future research direction is to examine whether this convergence behavior is theoretically guaranteed.

Overall, the qualitative behavior reported above matches our theoretical results in Section IV (e.g., the distance from  $\mathbf{p}^*$  is inversely proportional to  $\gamma$ ). We emphasize that the actual deviations from  $\mathbf{p}^*$  are much smaller than the theoretical guarantees. For example, the bounds in Theorem 4 can be an order of magnitude larger than the ones obtained in our experiments. This gap is quite expected, as our bounds are general, independent of the desired operating point and the actual system-utility. It remains a future research direction to tighten the bounds for specific cases of interest, e.g., by considering concrete system utilities and restricting the parameter space of the problem.

### D. Discussion

We briefly discuss here some consequences of our results, and also highlight some practical aspects. We have demonstrated in this section, both theoretically and empirically, that we can apply the so-called potential-game approach in order to enforce the network operating point to be close to a desired one. The appeal of the scheme lies in its generality, in the sense that any system objective can be (nearly) satisfied despite the self-interested nature of the underlying users.

The focus of this paper is on the game-theoretic analysis of a wireless network, where the prices are assumed to be set properly by a central network authority. Such central authority thus has to be equipped with full information on the parameters

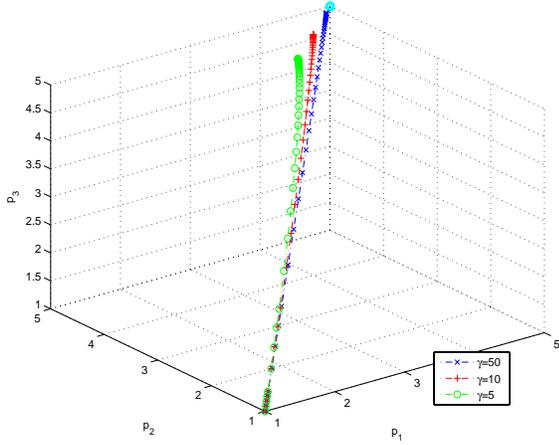


Fig. 1. The evolution of the power levels under best response dynamics. The starting point is  $\mathbf{p}^0 = [1, 1, 1]$ , the desired operating point is  $\mathbf{p}^* = [5, 5, 5]$ .

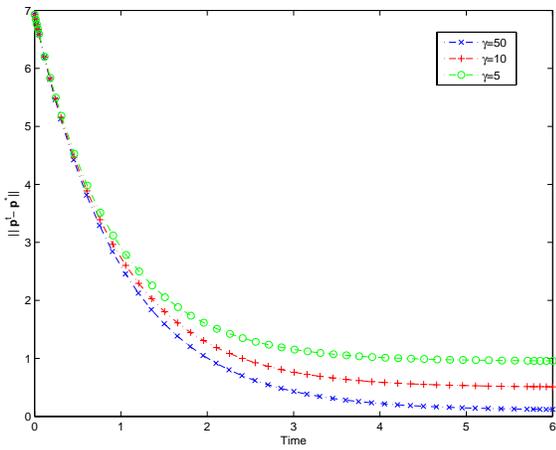


Fig. 2. The distance  $\|\mathbf{p}^t - \mathbf{p}^*\|$  between the current and desired power allocations, under best-response dynamics.

of the problem (such as the power constraints and the gain coefficients). In practice, however, such information may not be available centrally. A plausible way of dealing with this issue, is to set the prices themselves in a *distributed* manner, as we highlight below.

Assume that each base-station has complete knowledge of the parameters that affect the performance of its associated users (such as the cross-gain coefficients). Assume further that the base stations can communicate among themselves (e.g., through wired-based connections). Consider a system utility that is separable in  $m$ , i.e., of the form  $U_0(\mathbf{p}) = \sum_{m \in \mathcal{M}} U_{0,m}(\mathbf{p})$ , where  $U_{0,m}(\cdot)$  is the system-utility component associated with user  $m$  (the sum-rate objective is an example of such utility with  $U_{0,m}(\mathbf{p}) = r_m(\mathbf{p})$ ). Under this utility structure, one may consider the following two stage procedure for distributed price generation. At the first stage, the base stations exchange private information and jointly “agree” on a desired operating point (e.g., by using one of the distributed methods described in [22]). Consequently, each base-station sets the price for its associated mobile according

to (13). This two-stage procedure generates the prices  $\mathbf{c}^*$ , provided that the maximization of  $U_0(\mathbf{p})$  can be solved in a distributed manner (e.g., when  $U_{0,m}(\cdot)$  are concave functions). Otherwise (e.g., for non-concave system utilities) distributed price setting remains an open issue, since distributed optimization methods lack the tools to deal with such cases.

A final comment relates to possible inaccuracies in channel gain information. We have assumed throughout that the information about the channel gains is perfect, i.e., coincides with the true gains of the underlying network. Due to technological limitations, however, the estimated gains (i.e., the gains used for price setting) might differ from the true gains that determine the users’ throughputs (e.g., as a consequence of information quantization effects). In our framework, these differences translate to having different gains for the original game (the true gains) and the potentialized game (the estimated gains). Nonetheless, our analysis methodology can be extended to accommodate such inaccuracies. This direction will be formalized in future versions of the current work.

## V. THE SUM-RATE OBJECTIVE

We consider in this section the natural system objective of maximizing the sum rate in the network. The sum-rate objective can be formulated as maximization of the following utility

$$U_0(\mathbf{p}) = \sum_m r_m(\mathbf{p}). \quad (21)$$

We next apply Theorem 5 to obtain explicit bounds on the performance of BR dynamics.

*Theorem 6:* Let  $\mathbf{p}^*$  be the operating point that maximizes (21), and let  $\tilde{\mathcal{I}}_\epsilon$  be the set of  $\epsilon$ -equilibria to which BR dynamics converges (where  $\epsilon$  is given by (18)). Then

$$|U_0(\mathbf{p}^*) - U_0(\tilde{\mathbf{p}})| \leq \sqrt{2\epsilon}(M-1) \sum_{m \in \mathcal{M}} \frac{\bar{P}_m}{P_m} \quad (22)$$

for every  $\tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon$ .

*Proof:* (outline) The proof follows by bounding the partial derivatives of (21); we show in Lemma 2 (given in the appendix) that  $\left| \frac{\partial U_0}{\partial p_m} \right| \leq \frac{M-1}{P_m}$ . We then apply Theorem 5(ii), which immediately leads to (22). ■

We now examine through simulations the actual (sum-rate) performance of our scheme. Specifically, we are interested in both the temporal performance (i.e., the evolution of the sum-rate measure (21) in time), as well as the effect of  $\gamma$  on the overall deviation (in terms of the sum-rate) from the desired operating point.

We consider a network of ten users. We set  $N_0 = 1$ , the cross-gain coefficients  $h_{km}$  and the self-gain coefficients  $h_{mm}$  are chosen uniformly at random (from the intervals  $[0, 2]$  and  $[9, 11]$  respectively). The power constraints are identical for all players,  $\underline{P}_m = 10^{-2}$ ,  $\bar{P}_m = 10$  for all  $m \in \mathcal{M}$ . We consider different values of  $\gamma$  in the range  $\gamma \in [1, 100]$ . For each simulation instance, we solve for the desired operating point  $\mathbf{p}^*$  (the maximizer of (21)) and set the prices according to

(13). We note that the underlying optimization problem is non-convex and therefore (approximately) solved numerically by multiple executions of an optimization solver, each initialized at different starting points.

Figure 3 demonstrates the evolution of the sum-rate  $U_0$  in time for a typical simulation run (with  $\gamma = 10$ ). We initialize the dynamics at the minimal-power operating point  $[\underline{P}_1, \underline{P}_2 \dots \underline{P}_{10}]$ . We observe that the sum-rate in the system monotonously increases, obtaining a four-fold improvement in the sum-rate. From a practical perspective, this is an appealing property, since users, albeit selfish, keep increasing the system-utility, i.e., there is no degradation of performance at any point in time.

We conclude our experiments by examining the outcome of the best-response dynamics as a function of  $\gamma$ . As a concrete performance measure, we are interested in the (percentage) deviation of the obtained sum-rate from its optimal value. For a given  $t$  this measure is given by

$$100 \times \frac{U_0(\mathbf{p}^*) - U_0(\mathbf{p}^t)}{U_0(\mathbf{p}^*)}. \quad (23)$$

Since the dynamics is guaranteed to converge in finite time to a set, the long-run *average* deviation can be estimated by averaging (23) over  $t > T$  for large  $T$  (i.e., after the dynamics is confined in a small neighborhood of the desired operating point). Figure 4 depicts the (estimated long-run) average deviation as a function of  $\gamma$ . Two properties of the graph should be emphasized. First, the average deviation decreases with  $\gamma$ , as expected. More importantly, we see that even for small values of  $\gamma$ , the average deviation is quite small (around 3%). This indicates that despite setting the prices without considering the true utilities, the potential game approach leads to prices which induce near-optimal performance.

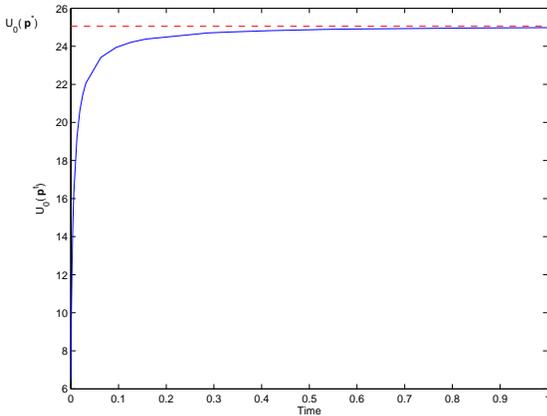


Fig. 3. The change in sum-rate as a function of time for  $\gamma = 10$ . The optimal sum-rate is marked by a dashed line for reference.

## VI. CONCLUDING REMARKS

This paper has considered the power control problem in CDMA wireless networks with self-interested wireless users.

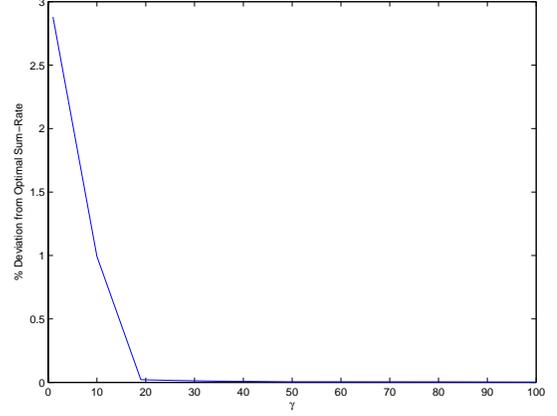


Fig. 4. The Effect of  $\gamma$  on performance loss. This graph depicts the (percentage) deviation of the obtained sum-rate from its optimal value, as a function of  $\gamma$ .

We have introduced the *potential-game approach* for distributed power allocation, which (approximately) enforces *any* power-dependent system-objective. This approach involves approximating the power game by a “close” potential game, for which target prices can be derived. By exploiting the relation between the power game and its approximation, the same prices induce *near-optimal* performance in the underlying system. Applying the potential-game approach for the CDMA wireless domain, we have established through Lyapunov-based analysis that best-response dynamics converges within a neighborhood of the desired operating point. We demonstrated through simulations that the actual performance could in practice be very close to optimal, even when operating at a low-SINR regime.

The scope of our work may be extended in several respects. As indicated in Section IV-D, the distributed implementation of pricing is possible, however we have not pursued in this paper any specific distributed method. The design and incorporation of distributed optimization schemes into our framework remains an important future research direction. In the network-pricing context, a challenging direction is to consider the case where the system is restricted to setting a user-independent price for power usage (which could be easier to implement in some configurations). It is of interest to examine the possible performance degradation due to this simpler pricing scheme.

On a higher level, our pricing scheme is made possible because the underlying user utilities form a game that is “close” to a potential game. In general noncooperative games, however, it is not apparent how to identify such a potential game with desirable properties. In an ongoing work [23], we develop a systematic procedure for choosing an appropriate potential game for any given game. We believe that the general paradigm of identifying “near-potential” games can be applied to other networking applications, thereby improving the controllability of networked systems with selfish users.

## REFERENCES

- [1] Z. Luo and S. Zhang, "Dynamic spectrum management: Complexity and duality," *IEEE Journal of Selected Topics in Signal Processing*, vol. 2, no. 1, pp. 57–73, 2008.
- [2] S. Hayashi and Z. Luo, "Spectrum management for interference-limited multiuser communication systems," *IEEE Transactions on Information Theory*, vol. 55, no. 3, pp. 1153–1175, 2009.
- [3] G. Scutari, D. Palomar, and S. Barbarossa, "Asynchronous iterative water-filling for gaussian frequency-selective interference channels," *IEEE Transactions on Information Theory*, vol. 54, no. 7, pp. 2868–2878, 2008.
- [4] F. Meshkati, H. Poor, and S. Schwartz, "Energy-efficient resource allocation in wireless networks," *IEEE Signal Processing Magazine*, vol. 24, no. 3, pp. 58–68, 2007.
- [5] M. Chiang, P. Hande, T. Lan, and C. Tan, "Power control in wireless cellular networks," in *Foundations and Trends in Networking, Volume 2, Issue 4*, 2008, pp. 381–533.
- [6] E. Altman and Z. Altman, "S-modular games and power control in wireless networks," *IEEE Transactions on Automatic Control*, vol. 48, no. 5, pp. 839–842, 2003.
- [7] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Transactions on Communications*, vol. 50, no. 2, pp. 291–303, 2002.
- [8] T. Alpcan, T. Basar, R. Srikant, and E. Altman, "CDMA uplink power control as a noncooperative game," *Wireless Networks*, vol. 8, no. 6, pp. 659–670, 2002.
- [9] T. Basar and R. Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," in *Proc. IEEE Infocom*, 2002, pp. 23–27.
- [10] L. He and J. Walrand, "Pricing and revenue sharing strategies for internet service providers," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 5, pp. 942–951, 2006.
- [11] D. Acemoglu and A. Ozdaglar, "Competition in parallel-serial networks," *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 6, pp. 1180–1192, 2007.
- [12] A. Ozdaglar and R. Srikant, "Incentives and pricing in communication networks," in *Algorithmic Game Theory*. Cambridge University Press, 2007.
- [13] J. Rosen, "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica*, vol. 33, no. 3, pp. 520–534, 1965.
- [14] D. Monderer and L. Shapley, "Potential Games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 124–143, 1996.
- [15] A. Neyman, "Correlated equilibrium and potential games," *International Journal of Game Theory*, vol. 26, no. 2, pp. 223–227, 1997.
- [16] D. Fudenberg and D. Levine, *The theory of learning in games*. The MIT Press, 1998.
- [17] J. Shamma and G. Arslan, "Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 312–327, 2005.
- [18] J. Hofbauer and W. Sandholm, "On the global convergence of stochastic fictitious play," *Econometrica*, pp. 2265–2294, 2002.
- [19] H. Khalil, *Nonlinear systems*. Prentice Hall, 2002.
- [20] O. Candogan, I. Menache, A. Ozdaglar, and P. Parrilo, "Near-optimal power control in wireless networks - a potential game approach," LIDS, MIT, Tech. Rep., available from <http://www.mit.edu/~candogan/publications/NetworkPricing.pdf>.
- [21] L. Andrew, "Measurement-based band allocation in multiband CDMA," *IEEE Transactions on Wireless Communications*, vol. 4, no. 2, pp. 475–483, 2005.
- [22] A. Nedic and A. Ozdaglar, "Cooperative distributed multi-agent optimization," in *Convex Optimization in Signal Processing and Communications*. Cambridge University Press, 2009.
- [23] O. Candogan, I. Menache, A. Ozdaglar, and P. Parrilo, "Near-Potential Games," (In preparation).

## ACKNOWLEDGMENT

This research was funded in part by National Science Foundation grants DMI-0545910 and ECCS-0621922, by the DARPA ITMANET program, and by a Marie Curie International Fellowship within the 7th European Community Framework Programme.

## APPENDIX

Throughout this section, we use the notation  $\lambda_m \triangleq \zeta_m c_m$  for all  $m \in \mathcal{M}$ .

*Proof of Lemma 1:* Let  $\bar{\phi} \triangleq \max_{\mathbf{p} \in \mathcal{P}} \phi(\mathbf{p})$  and  $\underline{\phi} \triangleq \min_{\mathbf{p} \in \mathcal{P}} \phi(\mathbf{p})$ , and define the function  $V = \bar{\phi} - \phi \geq 0$ . The idea behind the proof is to use  $V$  as a Lyapunov function for the analysis of the BR dynamics in  $\tilde{\mathcal{G}}$ .

Consider the time derivative of  $V$ , it is easy to see that

$$\begin{aligned} -\dot{V}(\mathbf{p}) &= \sum_{m \in \mathcal{M}} \frac{\partial \phi(\mathbf{p})}{\partial p_m} \dot{p}_m \\ &= \sum_{m \in \mathcal{M}} \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\beta_m(\mathbf{p}_{-m}) - p_m) \\ &= \sum_{m \in \mathcal{M}} \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\tilde{\beta}_m(\mathbf{p}_{-m}) - p_m) \\ &\quad + \sum_{m \in \mathcal{M}} \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\beta_m(\mathbf{p}_{-m}) - \tilde{\beta}_m(\mathbf{p}_{-m})). \end{aligned} \quad (24)$$

Note that by the strict concavity of the potential function it follows that

$$\begin{aligned} \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\tilde{\beta}_m(\mathbf{p}_{-m}) - p_m) \\ \geq \phi(\tilde{\beta}_m(\mathbf{p}_{-m}), \mathbf{p}_{-m}) - \phi(p_m, \mathbf{p}_{-m}) \\ = \tilde{u}_m(\tilde{\beta}_m(\mathbf{p}_{-m}), \mathbf{p}_{-m}) - \tilde{u}_m(p_m, \mathbf{p}_{-m}). \end{aligned} \quad (25)$$

We next obtain a lower bound on the term  $\sum_{m \in \mathcal{M}} \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\beta_m(\mathbf{p}_{-m}) - \tilde{\beta}_m(\mathbf{p}_{-m}))$ . Considering the bounds on the power investment of players, any  $\mathbf{p}^*$  satisfies  $\underline{P}_m \leq p_m^* \leq \bar{P}_m$ . In view of (13),  $\lambda_m$  satisfies

$$\underline{P}_m \leq \frac{1}{\lambda_m} \leq \bar{P}_m, \quad m \in \mathcal{M}. \quad (26)$$

Hence, it follows that

$$\left| \frac{\partial \phi(\mathbf{p})}{\partial p_m} \right| = \left| \frac{1}{p_m} - \lambda_m \right| \leq \frac{1}{\underline{P}_m} - \frac{1}{\bar{P}_m}. \quad (27)$$

For a given fixed  $\mathbf{p}_{-m}$  the unconstrained maximum of  $u_m$  with respect to  $p_m$  (denoted by  $q_m$ ) satisfies  $\frac{\partial u_m(q_m, \mathbf{p}_{-m})}{\partial p_m} = 0$ , where

$$\frac{\partial u_m(q_m, \mathbf{p}_{-m})}{\partial p_m} = \frac{\frac{\gamma h_{mm}}{N_0 + \sum_{k \neq m} h_{km} p_k}}{1 + \frac{\gamma h_{mm} q_m}{N_0 + \sum_{k \neq m} h_{km} p_k}} - \lambda_m.$$

Hence, solving for  $q_m$  we obtain,

$$q_m = \frac{1}{\lambda_m} - \frac{N_0}{\gamma h_{mm}} - \frac{1}{\gamma} \sum_{k \neq m} \frac{h_{km}}{h_{mm}} p_k.$$

It follows from (26) that  $q_m \leq \bar{P}_m$ ; using the concavity of  $u_m$ , it can be seen that the best response of player  $m$  satisfies

$$\beta_m(\mathbf{p}_{-m}) = \max \{ \underline{P}_m, q_m \}. \quad (28)$$

Similarly, it can be shown that for a fixed  $\mathbf{p}_{-m}$ , the unconstrained maximum of  $\tilde{u}_m$  with respect to  $p_m$  (denoted by  $\tilde{q}_m$ ) satisfies  $\frac{\partial \tilde{u}_m(\tilde{q}_m, \mathbf{p}_{-m})}{\partial p_m} = \frac{\partial \phi(\tilde{q}_m, \mathbf{p}_{-m})}{\partial p_m} = 0$ . Using the partial derivatives of the potential function, it follows that

$\tilde{q}_m = \frac{1}{\lambda_m}$ . By (26) it follows that the unconstrained maximum  $\tilde{q}_m$  always satisfies the power constraints, hence

$$\tilde{\beta}_m(\mathbf{p}_{-m}) = \frac{1}{\lambda_m}. \quad (29)$$

For every  $\mathbf{p}_{-m} \in \mathcal{P}_{-m}$  and  $m \in \mathcal{M}$ , (28)–(29) imply that  $\beta_m(\mathbf{p}_{-m}) \leq \tilde{\beta}_m(\mathbf{p}_{-m})$ , and furthermore

$$\begin{aligned} \tilde{\beta}_m(\mathbf{p}_{-m}) - \beta_m(\mathbf{p}_{-m}) &\leq \frac{N_0}{\gamma h_{mm}} + \frac{1}{\gamma} \sum_{k \neq m} \frac{h_{km}}{h_{mm}} p_k \\ &\leq \frac{N_0}{\gamma h_{mm}} + \frac{1}{\gamma} \sum_{k \neq m} \frac{h_{km}}{h_{mm}} \bar{P}_k = \frac{\xi_m}{\gamma}, \end{aligned} \quad (30)$$

where  $\xi_m \triangleq \frac{N_0}{h_{mm}} + \sum_{k \neq m} \frac{h_{km}}{h_{mm}} \bar{P}_k$  for every  $m \in \mathcal{M}$ . It thus follows from (27) and (30) that

$$\left| \frac{\partial \phi(\mathbf{p})}{\partial p_m} (\beta_m(\mathbf{p}_{-m}) - \tilde{\beta}_m(\mathbf{p}_{-m})) \right| \leq \left( \frac{1}{\underline{P}_m} - \frac{1}{\bar{P}_m} \right) \frac{\xi_m}{\gamma}.$$

By (24) and (25) it can be readily seen that  $V$  satisfies

$$\begin{aligned} -\dot{V}(\mathbf{p}) &\geq \sum_{m \in \mathcal{M}} \tilde{u}_m(\tilde{\beta}_m(\mathbf{p}_{-m}), \mathbf{p}_{-m}) - \tilde{u}_m(p_m, \mathbf{p}_{-m}) \\ &\quad - \sum_{m \in \mathcal{M}} \left( \frac{1}{\underline{P}_m} - \frac{1}{\bar{P}_m} \right) \frac{\xi_m}{\gamma} \end{aligned} \quad (31)$$

Therefore, (31) implies that  $\dot{V}(p) \leq -\sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}$  unless

$$\sum_{m \in \mathcal{M}} \tilde{u}_m(\tilde{\beta}_m(\mathbf{p}_{-m}), \mathbf{p}_{-m}) - \tilde{u}_m(p_m, \mathbf{p}_{-m}) \leq \sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}.$$

Note that  $\tilde{u}_m(\tilde{\beta}_m(\mathbf{p}_{-m}), \mathbf{p}_{-m}) - \tilde{u}_m(p_m, \mathbf{p}_{-m}) \geq 0$  for every  $\mathbf{p}_{-m}$  and  $m \in \mathcal{M}$ , because  $\tilde{\beta}_m(\mathbf{p}_{-m})$  is the maximizer of  $\tilde{u}_m$  when the users other than  $m$  are using  $\mathbf{p}_{-m}$ . Thus,  $\dot{V}(\mathbf{p}) \leq -\sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}$ , unless

$$\tilde{u}_k(\tilde{\beta}_k(\mathbf{p}_{-k}), \mathbf{p}_{-k}) - \tilde{u}_k(p_k, \mathbf{p}_{-k}) \leq \sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m} \quad (32)$$

for all  $k \in \mathcal{M}$ . Hence, with  $V$  as a Lyapunov function, we may apply a standard Lyapunov analysis argument to conclude that the dynamics in (15) converges to a set such that (32) holds for all  $k \in \mathcal{M}$ . Note that (32) implies that no player can improve its utility by more than  $\sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}$  by modifying its action. Thus, the dynamics converges to the set of  $\epsilon$ -equilibria ( $\tilde{\mathcal{I}}_\epsilon$ ) of the game  $\tilde{\mathcal{G}}$  where  $\epsilon \leq \sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}$ . By the definitions of  $\xi_m$  and  $\underline{SINR}_m$  it follows that  $\frac{\bar{P}_m}{\xi_m} = \underline{SINR}_m$ , hence

$$\epsilon \leq \frac{1}{\gamma} \sum_{m \in \mathcal{M}} \frac{1}{\underline{SINR}_m}.$$

The uniform convergence follows by noting that  $0 \leq V \leq \bar{\phi} + \underline{\phi}$ . Since the potential function has bounded partial derivatives and the joint strategy space is compact the latter term is bounded by some  $\bar{V}^4$ . Since  $\dot{V} \leq -\sum_{m \in \mathcal{M}} \frac{\xi_m}{\gamma \bar{P}_m}$  if the operating point is not in  $\tilde{\mathcal{I}}_\epsilon$ , the convergence time to  $\tilde{\mathcal{I}}_\epsilon$ ,

<sup>4</sup>Specifically, it can be shown that  $\bar{V} \leq \sum_{m \in \mathcal{M}} |\log \bar{P}_m| + |\log \underline{P}_m| + \lambda_m \bar{P}_m$  using the properties of the potential function.

which we denote by  $T$ , satisfies  $T \leq \frac{\gamma \bar{V}}{\sum_{m \in \mathcal{M}} \frac{\xi_m}{\bar{P}_m}}$  regardless of the initial operating point of the dynamics. Thus, the BR dynamics, converge uniformly to  $\tilde{\mathcal{I}}_\epsilon$ . ■

*Proof of Theorem 4:* The key idea behind the proof is to translate a deviation in the potential function value to a deviation in the strategy space  $\mathcal{P}$ , by using properties of the partial derivatives of the potential.

Since  $\tilde{\mathbf{p}} \in \tilde{\mathcal{I}}_\epsilon$ , it follows that  $\phi(p_m^*, \tilde{\mathbf{p}}_{-m}) - \phi(\tilde{p}_m, \tilde{\mathbf{p}}_{-m}) \leq \epsilon$ , or equivalently

$$(\log p_m^* - \lambda_m p_m^*) - (\log \tilde{p}_m - \lambda_m \tilde{p}_m) \leq \epsilon, \quad (33)$$

for every  $m \in \mathcal{M}$ . Let  $f_m : \mathcal{P}_m \rightarrow \mathbb{R}$  be a function such that

$$f_m(p_m) = (\log p_m - \lambda_m p_m).$$

Using second order expansion, it follows that

$$\begin{aligned} f_m(\tilde{p}_m) &= f_m(p_m^*) + (\tilde{p}_m - p_m^*) \frac{\partial f(p_m^*)}{\partial p_m} \\ &\quad + \frac{1}{2} (p_m^* - \tilde{p}_m)^2 \frac{\partial^2 f(p_m^* + \alpha(\tilde{p}_m - p_m^*))}{\partial p_m^2} \end{aligned}$$

for some  $\alpha \in [0, 1]$ . Note that  $\frac{\partial f(p_m^*)}{\partial p_m} = \frac{\phi(\mathbf{p}^*)}{\partial p_m} = 0$ , because  $\mathbf{p}^*$  is the desired operating point and all the partial derivatives of the potential vanish at this power allocation. Observing that  $\frac{\partial^2 f(p_m^*)}{\partial p_m^2} = -\frac{1}{p_m^2}$ , the previous equation can be rewritten as

$$f_m(p_m^*) - f_m(\tilde{p}_m) = \frac{1}{2} (p_m^* - \tilde{p}_m)^2 \frac{1}{(p_m^* + \alpha(\tilde{p}_m - p_m^*))^2},$$

or equivalently,

$$2(p_m^* + \alpha(\tilde{p}_m - p_m^*))^2 (f_m(p_m^*) - f_m(\tilde{p}_m)) = (p_m^* - \tilde{p}_m)^2.$$

Using (33) and the fact that  $0 < p_m^*, \tilde{p}_m \leq \bar{P}_m$ , the previous equation implies that  $2\epsilon \bar{P}_m^2 \geq (p_m^* - \tilde{p}_m)^2$ . We therefore conclude that  $\bar{P}_m \sqrt{2\epsilon} \geq |p_m^* - \tilde{p}_m|$ . ■

*Lemma 2:* The sum-rate function (21) satisfies  $|\frac{\partial U_0}{\partial p_m}| \leq \frac{M-1}{\bar{P}_m}$  for every  $m \in \mathcal{M}$ .

*Proof:* Differentiating  $U_0$  with respect to  $p_m$  we obtain

$$\frac{\partial U_0(\mathbf{p})}{\partial p_m} = \frac{\frac{\gamma h_{mm}}{N_0 + \sum_{k \neq m} h_{km} p_k}}{1 + \frac{\gamma h_{mm} p_m}{N_0 + \sum_{k \neq m} h_{km} p_k}} - \sum_{l \neq m} \frac{\frac{\gamma h_{ll} h_{ml} p_l}{(N_0 + \sum_{k \neq l} h_{kl} p_k)^2}}{1 + \frac{\gamma h_{ll} p_l}{N_0 + \sum_{k \neq l} h_{kl} p_k}}.$$

Thus it follows that

$$\frac{\partial U_0(\mathbf{p})}{\partial p_m} \leq \frac{\frac{\gamma h_{mm}}{N_0 + \sum_{k \neq m} h_{km} p_k}}{1 + \frac{\gamma h_{mm} p_m}{N_0 + \sum_{k \neq m} h_{km} p_k}} \leq \frac{1}{p_m} \leq \frac{1}{\bar{P}_m}$$

and

$$\begin{aligned} \frac{\partial U_0(\mathbf{p})}{\partial p_m} &\geq - \sum_{l \neq m} \frac{\frac{\gamma h_{ll} h_{ml} p_l}{(N_0 + \sum_{k \neq l} h_{kl} p_k)^2}}{1 + \frac{\gamma h_{ll} p_l}{N_0 + \sum_{k \neq l} h_{kl} p_k}} \\ &\geq - \sum_{l \neq m} \frac{h_{ml}}{(N_0 + \sum_{k \neq l} h_{kl} p_k)} \geq - \frac{M-1}{\bar{P}_m} \end{aligned}$$

for all  $\mathbf{p} \in \mathcal{P}$ . Therefore,

$$\left| \frac{\partial U_0}{\partial p_m} \right| \leq \frac{M-1}{\bar{P}_m}$$

■