Efficiency and Braess' Paradox under Pricing in General Networks

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***Abstract*— We study the flow control and routing decisions of self-interested users in a general congested network where a single profit-maximizing service provider sets prices for different paths in the network. We define an equilibrium of the user choices. We then define the monopoly equilibrium (ME) as the equilibrium prices set by the service provider and the corresponding user equilibrium. We analyze the networks containing different types of user utilities: elastic or inelastic. For a network containing inelastic user utilities, we show the flow allocations at the ME and the social optimum are the same. For a network containing elastic user utilities, we explicitly characterize the ME and study its performance relative to the user equilibrium at 0 prices and the social optimum that would result from centrally maximizing the aggregate system utility. We also define Braess' Paradox for a network involving pricing and show that Braess' Paradox does not occur under monopoly prices.**

***Index Terms*— Service provider, pricing, efficiency, Braess' paradox.**

I. Introduction

DESPITE the significant increase in bandwidth, management of congestion is still a major problem in communication networks. Such management typically involves two elements: flow control, i.e., the control of the amount of data sent by various users, and routing, i.e., the control of the route choices of data transmitted in the network. The standard approach to both flow control and routing is the regulation of network traffic in a centralized manner by a network manager (planner) with complete information about user needs and command over user actions, resulting in the so-called *system or social optimum*. Todays's networks emerged from interconnection of privately owned networks and serve heterogeneous users with different service needs. This motivated the need for the analysis of resource allocation in the presence of agents with multitude of economic interests and service requirements. Consequently, a recent theoretical literature considers a distributed control paradigm in which some network control functions are delegated to users and studies the selfish flow choices and routing behavior of users in the absence of central planning (see, among others, [2]-[10]). These models show that selfish behavior typically leads to allocations that are highly inefficient from a system point of view (e.g., too much flow or the wrong routing choices). The reason for this divergence between system optimum and user equilibrium is that users do not take into account the congestion that they cause for other users.

The fact that selfish behavior leads to inefficiency in performance has been well-recognized since the early work of Pigou [38]. There is a recent interest for quantifying this inefficiency, referred to as the *price of anarchy (POA)*, which is defined as the ratio of the performance of user equilibrium to the social optimum. In [34], Koutsoupias and Papadimitriou consider a two-link network with users that have fixed demands and study the performance of selfish routing. They provide a tight analysis of the ratio of the worst-case user equilibrium and the social optimum. The tight analysis of a parallel-link network with arbitrary number of links is given by Czumaj and Vöcking [35]. A recent paper by Roughgarden and Tardos [8] studies the POA for selfish routing for a general topology network. They show that when the latency functions are affine, the total latency of a user equilibrium is at most 4/3 of the minimum total latency (that is achieved at the social optimum). However, for more general convex latency functions, the total latency at the user equilibrium can be arbitrary large. The POA has also been studied for other types of resource allocation problems, such as resource allocation by market mechanisms (Johari and Tsitsiklis [10], Sanghavi and Hajek [42]) and network design (Anshelevich et. al. [43]).

In many real world networks, information is indeed decentralized and users are selfish, but they do also face prices and restrictions set by the service provider in the network. In most game-theoretic analysis of networking problems, pricing has been used as a means to cope with the inefficiency created by selfish users. In [4], Kelly shows that the network manager can use implicit prices (congestion signals) to induce the rate allocation that maximizes the total user utility. Similar results are given by Low and Lapsley in [5], Yaïche, Mazumdar, and Rosenberg in [36], and Korilis, Lazar, and Orda in [41]. There are many other works that study pricing as a tool to achieve efficiency. (see [36],[37] and the references therein) However, with a few exceptions ([7], [13], [14], [15]), the game-theoretic interaction between users and service providers have largely been neglected. In [7], Basar and Srikant analyze monopoly pricing under specific assumptions on the utility and latency functions. In [15], He and Walrand propose a fair revenue sharing scheme for multiple service providers under specific demand models. In [13], Acemoglu and Ozdaglar analyze equilibrium flows and routing in a parallel-link network and show how profit-maximizing prices from the viewpoint of the service provider typically also play the role of efficiently regulating data transmission.

In this paper, we analyze the equilibrium of a model that incorporates a self-interested service provider and study the performance gap between the equilibrium and the system optimum in a network with a general topology. Analysis of a general network is considerably more difficult than

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networks with parallel links. For a given price, we provide a characterization of the user equilibrium of flow rates and routing decisions under the standard Wardrop assumption that each user is small (thus ignores the effect of their decisions on aggregate congestion). Furthermore, we provide a full characterization of the "monopoly equilibrium", i.e., profit-maximizing prices from the viewpoint of service provider and the resulting allocations. We show that for the case of routing with participation control (see Sec. III, which naturally corresponds to the inelastic user utilities), the monopoly equilibrium achieves the system optimum. This result contrasts with pervasive inefficiencies in the routing models with selfish agents, for example, as in [8]. For the case of elastic user utilities, monopoly pricing introduces a distortion and induces users to reduce their flow rates. The performance of the monopoly equilibrium relative to a situation without prices and to the social optimal depends on the extent of the congestion effects (externalities). When these are important, the monopoly equilibrium, which forces users to internalize these effects, performs relatively well.

An important problem in general network topologies is the potential for network performance to deteriorate as a result of increasing network resources, which is also referred to as Braess' paradox [16]. A simple example of this is the possibility of the addition of a new link to increase congestion on all links in the network. Previous research has focused on the detection of Braess' paradox on specific network topologies and restrictions on methods of network upgrade for preventing it. We study the effects of profit-maximizing prices on Braess' paradox, and show that at the monopoly prices, there can never be Braess' paradox, so for-profit incentives appear sufficient to eliminate this type of paradoxical outcomes.

The rest of the paper is organized as follows. Section II describes the network topology and user preferences, provides the definition of a user equilibrium, and monopoly equilibrium. Section III shows the efficiency of the monopoly equilibrium in the case of users with inelastic utility. Section IV discusses the monopoly equilibrium in the case of users with elastic utility. It first analyzes the sensitivity of the equilibrium allocations to prices. Then, it defines and characterizes the monopoly equilibrium, and provides a comparison of the monopoly equilibrium with the social optimum. Finally, Section V discusses Braess' paradox under pricing.

II. Model: User Equilibrium, Monopoly Equilibrium, and Social Optimum

We consider a directed network $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ denotes the set of nodes and $\mathcal{E}$ denotes the set of links. We assume that there are m origin-destination (OD) node pairs $\{s_1, t_1\}, ..., \{s_m, t_m\}$, and we denote the set of OD pairs by $\mathcal{W}$. For each OD pair $\{s_k, t_k\} \in \mathcal{W}$, there are J_k users, belonging to set $\mathcal{J}_k$, that send data from node s_k to node t_k through paths that connect s_k and t_k. We also denote the set of paths between s_k and t_k by $\mathcal{P}_k$ and the set of all paths in the network by $\mathcal{P} = \cup_{k\in\mathcal{W}}\mathcal{P}_k$. We say a link $e \in p$ when the link lies along the path p.

To facilitate our analysis, we first introduce some of the notations that we will use in the discussion. We denote $f^p_{k,j}$ to be the flow[1] of user j of OD pair k on path p where $j \in \mathcal{J}_k$ and $p \in \mathcal{P}_k$. We then use

$$f^p = \sum_{j\in\mathcal{J}_k} f^p_{k,j}$$

to represent the total flow on path p and

$$f^e = \sum_k \sum_{j\in\mathcal{J}_k} \sum_{\{p|e\in p, p\in\mathcal{P}_k\}} f^p_{k,j} = \sum_k \sum_{\{p|e\in p, p\in\mathcal{P}_k\}} f^p \quad (1)$$

to represent the total flow (link load) on link e. We also use the following notation to represent different flow vectors:

$$\boldsymbol{g} \;:\; [f^e]_{e\in\mathcal{E}}, \text{vector of link loads.} \quad (2)$$

$$\boldsymbol{h} \;:\; [f^p]_{p\in\mathcal{P}}, \text{vector of path flows.} \quad (3)$$

$$\boldsymbol{f}_k \;:\; [f^p]_{p\in\mathcal{P}_k}, \text{vector of path flows of OD pair } k. \quad (4)$$

$$\boldsymbol{f}_{k,j} \;:\; [f^p_{k,j}]_{p\in\mathcal{P}_k}, \text{vector of path flows of user } j. \quad (5)$$

$$\boldsymbol{f} \;:\; [f^p_{k,j}]_{p\in\mathcal{P}_k, j\in\mathcal{J}_k, k\in\mathcal{W}}, \text{vector of flows of all users.} \quad (6)$$

Finally, we denote

$$\Gamma_{k,j} = \sum_{p\in\mathcal{P}_k} f^p_{k,j}.$$

to be the total flow rate of user j.

In the absence of central regulation, we assume that each user in the network is interested in his own payoff. This payoff should reflect the tradeoff between the utility of sending data and the disutility of incurred delays and monetary costs during transmission. We next formalize the user payoff function.

We assume each user $j \in \mathcal{J}_k$ receives a utility of $u_{k,j}(\Gamma_{k,j})$. Depending on the application service requirements, the utility function takes different forms. Shenker [9] categorized applications into two main classes based on their service requirements: *inelastic and elastic applications*. Real-time voice and video applications require a fixed amount of bandwidth for adequate QoS, hence are inelastic in their demand for bandwidth. Therefore, it is reasonable to model their utility as a step function, see Figure 1(a). On the other hand, traditional applications such as e-mail and file transfer are elastic; they are tolerant of delay and can take advantage of even the minimal amounts of bandwidth. The utility function in this case can be represented as a nondecreasing and concave function, see Figure 1(b). We assume that each user is using only one application. A user who is using multiple applications can be viewed as multiple users, each using one application. Different users might have different utility functions even though they are using the same type of application, representing different preferences. We say that a user with an inelastic (elastic) application has an inelastic (elastic) utility function. Both utility classes can be analyzed within the framework introduced here.

To model delays incurred during transmission, we assume that each link e has a flow-dependent latency function $l^e(f^e)$ where f^e is link load on link e [cf. Eq. (1)]. The latency cost of sending one unit of flow on path p is then given by

$$\sum_{e\in p} l^e(f^e) \quad (7)$$

[1]We use the term flow to represent the data stream that the user sends.

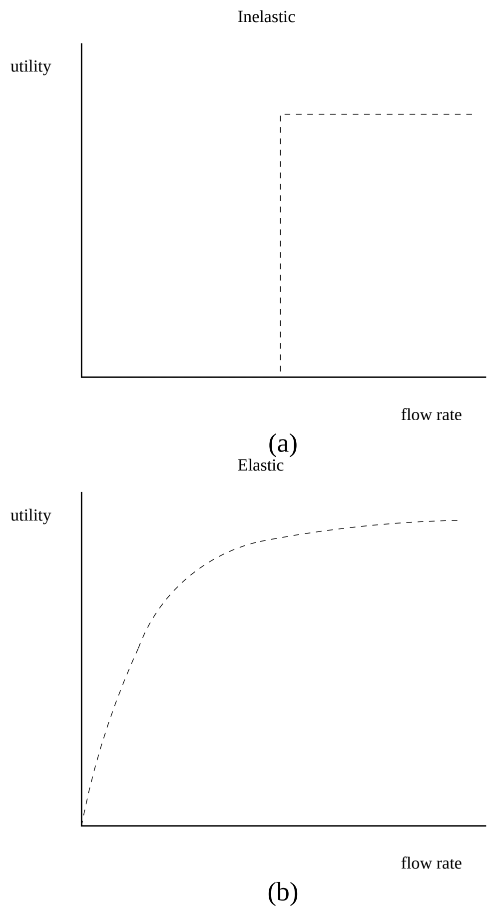


Fig. 1. a) Inelastic utility as a function of flow rate. b) Elastic utility as a function of flow rate.

and the latency of sending $f_{k,j}^p$ units of flow along path p is given by

$$\sum_{e \in p} l^e(f^e) f_{k,j}^p.$$

The additive latency cost is an assumption that is used extensively in communication and transportation literature. In practice, the end-to-end delay encountered by the flows may depend on other factors than the link loads. For example, in the Internet, there is a processing delay on each node associated with the total flow entering the node (see [41]). The end-to-end delay may also have more complicated structures than the additive structure defined in Eq. (7). Nevertheless, our model provides a tractable framework for capturing the essential aspects of queueing delay and is a good approximation to delay costs in real networks (see [39]). [2]

For the cost of services, we assume that the service provider charges a price q^p per unit of bandwidth for path p. We denote $\boldsymbol{q}$ to be the price vector $[q^p]_{p \in \mathcal{P}}$. Given the prices set by the service provider, the goal of each user in the network is to maximize his own payoff. Note that an alternative model is one in which the service provider charges prices for links rather than paths. However, it can be seen that the service provider can make more profit by charging prices for the paths.

[2]Qiu et. al. have discussed some representative link latency functions for the Internet in [39].

We will adopt the following assumptions on utility and link latency functions.

Assumption 1: Assume that for each user $j \in \mathcal{J}_k$, the utility function $u_{k,j}$ is nondecreasing. For elastic user utility functions, we further assume that the functions are strictly concave, continuously differentiable, and $0 < u'_{k,j}(0) < \infty$. We also assume that for each link e, the latency function l^e is continuous and strictly increasing.

We next define the user payoff function: For a given price q, each user j chooses his path flows $\boldsymbol{f}_{k,j}$ to maximize his payoff function

$$v_{k,j}(\boldsymbol{f}_{k,j}; \boldsymbol{g}, \boldsymbol{q}) = u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f_{k,j}^p - \sum_{p \in \mathcal{P}_k} q^p f_{k,j}^p. \quad (8)$$

where $\boldsymbol{g}$ is defined in (2).

As is common in traffic equilibrium models used in transportation and communication networks, we assume that each user is small, thus focus on Wardrop Equilibria, where the individual user does not anticipate the effect of his flow on the total level of congestion. [1, 8, 13] This appears as a realistic assumption in today's large-scale data networks such as the Internet and transportation networks. Standard arguments establish that Wardrop equilibria are obtained as estimates of Nash equilibria as the number of users go to ∞ (see [18]).

Definition 1: Let $\boldsymbol{f}$ be the vector of flows of all users in the network that is defined in (6). For a given price vector $\boldsymbol{q} \geq 0$, a flow vector $\boldsymbol{f}^*$ is a Wardrop equilibrium (WE) of the user game if

$$\boldsymbol{f}_{k,j}^* \in \arg\max_{\boldsymbol{f}_{k,j} \geq 0} v_{k,j}(\boldsymbol{f}_{k,j}; \boldsymbol{g}, \boldsymbol{q}), \ \forall\ j \in \mathcal{J}_k, k \in \mathcal{W},$$

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p | e \in p, p \in \mathcal{P}_k} (f^*)_{k,j}^p, \qquad \forall\ e \in \mathcal{E}.$$

Hence, each price vector induces a WE among the users. The service provider (monopolist) chooses the price vector to maximize his profit. The maximization problem can be written as:

$$\max_{\boldsymbol{q} \geq 0} \quad \sum_p q^p f^p(\boldsymbol{q}), \quad (9)$$

where $f^p(\boldsymbol{q})$ is the flow on path p at a WE for a given price vector $\boldsymbol{q}$. We will show in later sections that under Assumption (1), Problem (9) has an optimal solution, which we denote by $\boldsymbol{q}^*$. We refer to $\boldsymbol{q}^*$ as the *monopoly price*. Let $\boldsymbol{f}^* = \boldsymbol{f}(\boldsymbol{q}^*)$ be the flow vector at a WE for price $\boldsymbol{q}^*$. Then we call $(\boldsymbol{q}^*, \boldsymbol{f}^*)$ the *monopoly equilibrium* (ME) of the problem.

To study the performance of the ME, we compare the total system utility at the equilibrium with the total system utility at the network's *social optimum*. A flow $\boldsymbol{f}$ is a social optimum if it maximizes the total system utility:

$$\sum_{k \in \mathcal{W}} \sum_{j \in \mathcal{J}_k} \left(u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f_{k,j}^p \right). \quad (10)$$

We can view the social optimum as the allocation that would be chosen by a network planner, which has full information and control over the network. The allocation at an ME is not necessarily the same as the social optimum. In the following, we analyze the performance of the ME relative to the social optimum for both inelastic and elastic user utilities. The different structure of the utility functions introduces significant differences in the analysis and the resulting performances of these utility classes.

III. Inelastic User Utility (Routing with Participation Control)

We first analyze a network containing users with inelastic utility functions. When a user has an inelastic utility function, it can be seen from Eq. (8) that at a given price vector, he either sends a fixed amount of data or decides not to participate in the network. Hence, the problem with inelastic utility functions is a routing problem, where user j is interested in choosing the paths to send his fixed amount of data, say $t_{k,j}$ units; but he also has the option of not sending any data when it is costly to do so. This is also a natural model to study the routing problem in the presence of service providers since it prevents the service provider from charging infinite prices. We refer to this problem as *the routing problem with participation control*. This problem was studied for parallel link networks in [13]. Here, we extend this analysis to general networks.

The problem can be modelled using the following utility function for user j

$$u_{k,j}(x) = \begin{cases} 0, & \text{if } 0 \le x < t_{k,j}, \\ t_{k,j}, & \text{if } x \ge t_{k,j}, \end{cases} \qquad (11)$$

together with binary variables $z_{k,j}$ which indicate whether user j chooses to participate or not, i.e., $z_{k,j} = 1$ if user j decides to send $t_{k,j}$ units of data, and $z_{k,j} = 0$ if he decides not to send any data. Denote $\boldsymbol{z}$ to be the vector $[z_{k,j}]_{j\in\mathcal{J}_k,k\in\mathcal{W}}$. The user equilibrium of this problem can be defined as follows.

Definition 2: For a given price vector $\boldsymbol{q} \ge 0$, a vector $(\boldsymbol{f}^*, \boldsymbol{z}^*)$, is a WE of the routing problem with participation control if for all k and all $j \in \mathcal{J}_k$,

$$(\boldsymbol{f}^*_{k,j}, z^*_{k,j}) \in \arg \max_{\boldsymbol{f}_{k,j}\ge 0, z_{k,j}\in\{0,1\}} \left\{ u_{k,j}(\Gamma_{k,j} z_{k,j}) - \sum_{p\in\mathcal{P}_k} \left(\sum_{e\in p} (l^e(f^e)) + q^p \right) f^p_{k,j} \right\}, \qquad (12)$$

$$f^e = \sum_k \sum_{j\in\mathcal{J}_k} \sum_{p|e\in p, p\in\mathcal{P}_k} (f^*)^p_{k,j}, \quad \forall\, e \in \mathcal{E},$$

where $u_{k,j}$ is given by Eq. (11).

Since the utility function [Eq. (11)] is not concave, we cannot guarantee the existence of a WE for any price vector. In fact, a WE may not exist for some price vectors. For example, consider a network that consists of one directed link where two users, A and B, send data through this link. Assume that $t_A = 1,\ t_B = 1.5,\ l(x) = \frac{1}{2}x$. It can be seen that if the price of the link is 0, a WE does not exist. In the same example, however, one can also show that the profit-maximizing price set by the monopolist is 0.5, and at this price, there exists a WE in which A sends his data and B does not. In the following, we show that at the monopoly price, there exists a WE, which moreover achieves the social optimum. (i.e., the flow allocations at any ME and the social optimum are the same). For consistency, we define the social optimum for the inelastic utility case as a vector $(\boldsymbol{f}, \boldsymbol{z})$ that maximizes the total system utility:

$$\sum_{k\in\mathcal{W}} \sum_{j\in\mathcal{J}_k} \left(u_{k,j}(\Gamma_{k,j} z_{k,j}) - \sum_{p\in\mathcal{P}_k} \left(\sum_{e\in p} l^e(f^e) \right) f^p_{k,j} \right). \qquad (13)$$

Proposition 1: Consider a routing problem with participation control.

1) *There exists a monopoly price $\boldsymbol{q}$, and a WE $(\boldsymbol{f}, \boldsymbol{z})$ at price $\boldsymbol{q}$.*
2) *A vector $(\boldsymbol{f}, \boldsymbol{z})$ is a social optimum if and only if there exists a price vector $\boldsymbol{q}$ such that $(\boldsymbol{q}, (\boldsymbol{f}, \boldsymbol{z}))$ is an ME.*

Proof: To establish this proposition, we first prove two lemmas. The first lemma gives a characterization of a WE at any price vector and the second one gives an explicit characterization of the monopoly price. The first lemma is proved by exploiting the linear structure of problem (12).

Lemma 1: For a given price vector $\boldsymbol{q} \ge 0$, a vector $(\boldsymbol{f}, \boldsymbol{z})$, with $\boldsymbol{f}_{k,j} \ge 0,\ z_{k,j} \in \{0, 1\}\ \forall\ k, j$, is a WE if and only if it satisfies the following conditions:

1) $f^e = \sum_k \sum_{j\in\mathcal{J}_k} \sum_{p|e\in p, p\in\mathcal{P}_k} f^p_{k,j}, \quad \forall\ e \in \mathcal{E}$.
2) *If $z_{k,j} = 1$, $\sum_{p\in\mathcal{P}_k} f^p_{k,j} = t_{k,j}$.*
3) *If $z_{k,j} = 0$, $f^p_{k,j} = 0$ for all $p \in \mathcal{P}_k$.*
   Define the set

$$\overline{\mathcal{P}_k} = \left\{ p \mid p \in \mathcal{P}_k \text{ and } \sum_{e\in p} l^e(f^e) + q^p \le \min\{1, \min_{m\in\mathcal{P}_k} \{ \sum_{e\in m} l^e(f^e) + q^m \}\} \right\}.$$

4) *If $p \notin \overline{\mathcal{P}_k}$, then $f^p_{k,j} = 0,\ \forall\ j \in \mathcal{J}_k$.*
5) *If $\min_{m\in\mathcal{P}_k}\{\sum_{e\in m} l^e(f^e) + q^m\} < 1$, then $z_{k,j} = 1$ for all $j \in \mathcal{J}_k$ and k.*

Proof: The proof of the necessity of conditions (1) - (5) is immediate. We show that these conditions are sufficient. We rewrite problem (12) as:

$$(\boldsymbol{f}^*_{k,j}, z^*_{k,j}) \in \arg \max_{\boldsymbol{f}_{k,j}\ge 0, z_{k,j}\in\{0,1\}} \left\{ t_{k,j} z_{k,j} - \sum_{p\in\mathcal{P}_k} \left(\sum_{e\in p} l^e(f^e) + q^p \right) f^p_{k,j} \right\} \qquad (14)$$

$$\text{s.t.} \quad \sum_{p\in\mathcal{P}_k} f^p_{k,j} = t_{k,j},\ if\ z_{k,j} = 1. \qquad (15)$$

Let $(\bar{\boldsymbol{f}}, \bar{\boldsymbol{z}})$ be a vector satisfying conditions (1) - (5). To show that this vector is a WE, we show that for all j,

$$\left\{t_{k,j}\bar{z}_{k,j} - \sum_{p\in\mathcal{P}_k}\left(\sum_{e\in p} l^e(f^e) + q^p\right)\bar{f}^p_{k,j}\right\}$$
$$> \quad \left\{t_{k,j}z_{k,j} - \sum_{p\in\mathcal{P}_k}\left(\sum_{e\in p} l^e(f^e) + q^p\right) f^p_{k,j}\right\}, \quad (16)$$

where

$$f^e = \sum_k \sum_{j\in\mathcal{J}_k} \sum_{p|e\in p, p\in\mathcal{P}_k} \bar{f}^p_{k,j}, \quad \forall\ e\in\mathcal{E},$$

and $(\boldsymbol{f}_{k,j}, z_{k,j})$ is any feasible solution of problem (14). Note that the f^e on the both sides of inequality (16) are the same for all e since each user is small and does not anticipate the effect of his flow on the total level of congestion. Now, we consider an arbitrary user $j\in\mathcal{J}_k$. There are two cases:

Case 1: $z_{k,j} \neq \bar{z}_{k,j}$.

First consider the case $\bar{z}_{k,j} = 0$ and $z_{k,j} = 1$. By condition (3), $\bar{z}_{k,j} = 0$ implies that $\bar{f}^p_{k,j} = 0\ \forall\ p$. Therefore, user j's payoff is 0 at $(\bar{\boldsymbol{f}}_{k,j}, \bar{z}_{k,j})$. By condition (5), we further have

$$\min_{m\in\mathcal{P}_k}\left\{\sum_{e\in m} l^e(f^e) + q^m\right\} \geq 1.$$

Since $z_{k,j} = 1$, this shows that user j's payoff is less than or equal to 0 at $(\boldsymbol{f}_{k,j}, z_{k,j})$. Next assume that $\bar{z}_{k,j} = 1$ and $z_{k,j} = 0$. Condition (4) implies that user j's payoff is greater than or equal to 0 at $(\bar{\boldsymbol{f}}_{k,j}, \bar{z}_{k,j})$. However, $z_{k,j} = 0$ implies by problem (14) that user j's payoff is less than or equal to 0 at $(\boldsymbol{f}_{k,j}, z_{k,j})$. Therefore, for both cases, user j's payoff at $(\bar{\boldsymbol{f}}_{k,j}, \bar{z}_{k,j})$ is greater than or equal to his payoff at $(\boldsymbol{f}_{k,j}, z_{k,j})$.

Case 2: $z_{k,j} = \bar{z}_{k,j}$.

For the case where $z_{k,j} = \bar{z}_{k,j} = 0$, user j's payoff is 0 at $(\bar{\boldsymbol{f}}_{k,j}, \bar{z}_{k,j})$ [cf. condition (3)] and is less than or equal to 0 at $(\boldsymbol{f}_{k,j}, z_{k,j})$. Next, we look at the case where $z_{k,j} = \bar{z}_{k,j} = 1$. By condition (4), it follows that for all paths p such that $\bar{f}^p_{k,j} > 0$, we have

$$\sum_{e\in p} l^e(f^e) + q^p = \min\left\{1, \min_{m\in\mathcal{P}_k}\{\sum_{e\in m} l^e(f^e) + q^m\}\right\}.$$

In view of the linear structure of the problem, this shows that user j's payoff at $(\bar{\boldsymbol{f}}_{k,j}, \bar{z}_{k,j})$ is greater than or equal to his payoff at $(\boldsymbol{f}_{k,j}, z_{k,j})$. **Q.E.D.**

For the second lemma, we consider the monopoly problem for the routing problem with participation control,

$$\begin{aligned}
\max \quad & \sum_{p\in\mathcal{P}} q^p f^p \qquad (17)\\
\text{subject to} \quad & f^p = \sum_{j\in\mathcal{J}_k} f^p_{k,j}, \quad \forall\ p\in\mathcal{P},\\
& \boldsymbol{q} \geq \boldsymbol{0},\\
& (\boldsymbol{f}, \boldsymbol{z}) \in G(\boldsymbol{q}),
\end{aligned}$$

where $G(\boldsymbol{q})$ is the set of vectors $(\boldsymbol{f}, \boldsymbol{z})$ that satisfy conditions (1)-(5) of Lemma 1.

Lemma 2: Let $(\boldsymbol{q}, (\boldsymbol{f}, \boldsymbol{z}))$ be an ME. Then, for all p with $f^p > 0$, we have

$$q^p = 1 - \sum_{e\in p} l^e(f^e). \qquad (18)$$

Proof: Since $(\boldsymbol{f}, \boldsymbol{z})$ is a feasible solution of problem (17), $(\boldsymbol{f}, \boldsymbol{z})$ is a WE. Let p be a path in $\mathcal{P}_k$ with positive flow ($f^p > 0$). By condition (4) in Lemma 1, we have $p\in\overline{\mathcal{P}_k}$. Therefore, by condition (3) we have

$$q^p + \sum_{e\in p} l^e(f^e) \leq 1.$$

Now, assume $q^p + \sum_{e\in p} l^e(f^e) < 1$, then for every $p'\in\mathcal{P}_k$ with $f^{p'} > 0$ we have

$$\begin{aligned}
q^p + \sum_{e\in p} l^e(f^e) &= q^{p'} + \sum_{e\in p'} l^e(f^e)\\
&< \min\left\{1, \min_{m\notin\overline{\mathcal{P}_k}} \sum_{e\in m} l^e(f^e) + q^m\right\},\\
&\quad \forall\ p'\in\overline{\mathcal{P}_k}.
\end{aligned}$$

Hence, there exists some $\epsilon > 0$ such that

$$\begin{aligned}
& q^{p'} + \sum_{e\in p'} l^e(f^e) + \epsilon <\\
&\quad \min\left\{1, \min_{m\notin\overline{\mathcal{P}_k}} \sum_{e\in m} l^e(f^e) + q^m\right\},\ \forall\ p'\in\overline{\mathcal{P}_k}.
\end{aligned}$$

Now, let $\boldsymbol{q}' = \boldsymbol{q} + \boldsymbol{e}_m$, where $\boldsymbol{e}_m$ is a $|\mathcal{P}|$-dimensional vector with value ϵ in the m^{th} component if $m\in\overline{\mathcal{P}_k}$, and 0 otherwise. We can verify that, given price vector $\boldsymbol{q}'$, $(\boldsymbol{f}, \boldsymbol{z})$ satisfies all of the conditions in Proposition 1. Therefore, $(\boldsymbol{f}, \boldsymbol{z})$ is a WE at price $\boldsymbol{q}'$. However, $(\boldsymbol{q}', (\boldsymbol{f}, \boldsymbol{z}))$ has a strictly higher objective value than $(\boldsymbol{q}, (\boldsymbol{f}, \boldsymbol{z}))$, which contradicts the fact that $(\boldsymbol{q}, (\boldsymbol{f}, \boldsymbol{z}))$ is an ME. Therefore, $q^p = 1 - \sum_{e\in p} l^e(f^e)$ for every p with $f^p > 0$. Now if $f^p = 0$, condition (5) implies $q^p \geq 1 - \sum_{e\in p} l^e(f^e)$. **Q.E.D.**

We now return to the proof of Proposition 1. We first consider the following problem.

$$\begin{aligned}
\max \quad & \sum_{p\in\mathcal{P}} q^p f^p \qquad (19)\\
\text{subject to} \quad & f^p = \sum_{j\in\mathcal{J}_k} f^p_{k,j}, \quad \forall\ p\in\mathcal{P},\\
& q^p = 1 - \sum_{e\in p} l^e(f^e),\ \text{if}\ f^p = 0,\\
& q^p \geq 0,\ if\ f^p > 0,\\
& (\boldsymbol{f}, \boldsymbol{z}) \in G(\boldsymbol{q}),
\end{aligned}$$

where $G(\boldsymbol{q})$ is the set of vectors $(\boldsymbol{f}, \boldsymbol{z})$ that satisfy conditions (1)-(5) of Lemma 1. It can be shown that $(\boldsymbol{q}, (\boldsymbol{f}, \boldsymbol{z}))$ is an optimal solution of problem (17) if and only if there exists a price $\bar{\boldsymbol{q}}$ such that $(\bar{\boldsymbol{q}}, (\boldsymbol{f}, \boldsymbol{z}))$ is an optimal solution of problem (19).

Now, we can rewrite problem (19) as

$$\begin{aligned} \max \quad & \sum_{p\in\mathcal{P}}(1-\sum_{e\in p} l^e(f^e))f^p \\ \text{subject to} \quad & f^p = \sum_{j\in\mathcal{J}_k} f^p_{k,j}, \quad \forall\, p\in\mathcal{P} \\ & \sum_{p\in\mathcal{P}_k} f^p_{k,j} = t_{k,j},\ \text{if } z_{k,j}=1, \\ & f^p_{k,j}=0,\ \forall\, p\in\mathcal{P}_k,\ \text{if } z_{k,j}=0, \\ & \boldsymbol{f}_{k,j}\geq 0,\ z_{k,j}\in\{0,1\},\ \forall\, j\in\mathcal{J}_k, \forall\, k, \end{aligned}$$

or equivalently,

$$\begin{aligned} \max_{\boldsymbol{f}_{k,j}\geq 0, z_{k,j}\in\{0,1\}} \quad & \sum_k \sum_{j\in\mathcal{J}_k}\Big(z_{k,j}t_{k,j} - \sum_{p\in\mathcal{P}_k}\sum_{e\in p} l^e(f^e) f^p_{k,j}\Big) \qquad (20) \\ \text{subject to} \quad & f^p = \sum_{j\in\mathcal{J}_k} f^p_{k,j}, \quad \forall\, p\in\mathcal{P}, \\ & \sum_{p\in\mathcal{P}_k} f^p_{k,j} = t_{k,j},\ \text{if } z_{k,j}=1 \\ & \boldsymbol{f}_{k,j}\geq 0,\ z_{k,j}\in\{0,1\},\ \forall\, j\in\mathcal{J}_k, \forall\, k. \end{aligned}$$

This problem has an optimal solution (since for each $\boldsymbol{z}$, the objective function is continuous and the constraint set is compact). This proves part (1) of Proposition 1. For part (2), we notice that problem (20) is the same as the social problem that maximizes the aggregate system utility as defined by Eq. (13). Hence, the result in part (2) of Proposition 1 follows. **Q.E.D.**

IV. Elastic User Utility

In this section, we study a network containing users with elastic utility functions.

A. Existence, Essential Uniqueness, and Price Sensitivity

Each price vector $\boldsymbol{q}$ defines a user subgame. Given the price vector, users play this subgame by choosing the flow rates and path flows that maximize their payoffs. If a WE exists, then at this WE, no user can increase his payoff by any deviation, so he does not have any incentive to deviate. We make a further assumption on link latency functions:

Assumption 2: Assume $l^e(f^e)\to\infty$ as $f^e\to C^e$, where C^e denotes the available capacity on link e.

This assumption on the latency functions serves to guarantee that no individual has an infinite demand. This assumption could be relaxed by assuming that, for each j, there exists a nonzero scalar B_j such that $u'_j(B_j)=0$, which holds for the inelastic utility case.

Proposition 2: **(Existence-Essential Uniqueness)** *Let Assumptions (1) and (2) hold. For a given price vector $\boldsymbol{q}$, let the payoff function for each user in the network be defined as Eq. (8). Then for any $\boldsymbol{q}\geq\boldsymbol{0}$, the user game has a WE. Moreover, the user flow rates and link loads at any WE are unique.*

The proof uses standard arguments used in transportation and communication networks literature (see [19,20]), and is therefore omitted. Essential uniqueness of a WE is important for our analysis, since it implies that total flows on each path are uniquely defined. This result does not, however, imply the uniqueness of a WE. In fact, it is easy to establish that when there is one OD pair with at least two users with positive equilibrium flows and at least two paths with positive total flows, then there are infinitely many WEs.[3] We use this property of a WE in proving the following result, which will be essential in our subsequent analysis.

Lemma 3: Let Assumptions (1) and (2) hold. Given any price $\boldsymbol{q}\geq\boldsymbol{0}$, let $\boldsymbol{f}$ be a WE, and Γ be the flow rate at price $\boldsymbol{q}$. Let f^p be the flow on path p. Also define $\overline{\mathcal{P}_k}=\{p\mid f^p>0,\ p\in\mathcal{P}_k\}$ and $\overline{\mathcal{J}_k}=\{j\mid \Gamma_{k,j}>0,\ j\in\mathcal{J}_k\}$ for every k. Then

1) *For every k, if $p\in\overline{\mathcal{P}_k}$ and $j\in\overline{\mathcal{J}_k}$,*
$$u'_{k,j}(\Gamma_{k,j})-\sum_{e\in p} l^e(f^e)-q^p=0.$$
2) *There exists a WE $\boldsymbol{f}$ such that $f^p_{k,j}>0$ for all $p\in\overline{\mathcal{P}_k}$, $j\in\overline{\mathcal{J}_k}$, and for all k.*

Proof: 1) Let $p\in\overline{\mathcal{P}_k}$, and $j\in\overline{\mathcal{J}_k}$. Since $\Gamma_{k,j}>0$, there exists some path s such that $f^s_{k,j}>0$, which implies by the first order conditions that

$$u'_{k,j}(\Gamma_{k,j})-\sum_{e\in s} l^e(f^e)-q^s=0 \qquad (21)$$

and

$$u'_{k,j}(\Gamma_{k,j})-\sum_{e\in s'} l^e(f^e)-q^{s'}\leq 0,\ \forall\, s'\in\mathcal{P}_k.$$

Combining the preceding two relations, we obtain

$$\sum_{e\in s'} l^e(f^e)+q^{s'}\geq\sum_{e\in s} l^e(f^e)+q^s,\ \forall\, s'\in\mathcal{P}_k.$$

Therefore,

$$\sum_{e\in s} l^e(f^e)+q^s=\min_{s'\in\mathcal{P}_k}\left\{\sum_{e\in s'} l^e(f^e)+q^{s'}\right\}. \qquad (22)$$

Now, since $f^p>0$, there exists some j' such that $f^p_{k,j'}>0$. Then,

$$u'_{k,j'}(\Gamma_{k,j'})-\sum_{e\in p} l^e(f^e)-q^p=0$$

and

$$u'_{k,j'}(\Gamma_{k,j'})-\sum_{e\in s'} l^e(f^e)-q^{s'}\leq 0,\ \forall\, s'\in\mathcal{P}_k.$$

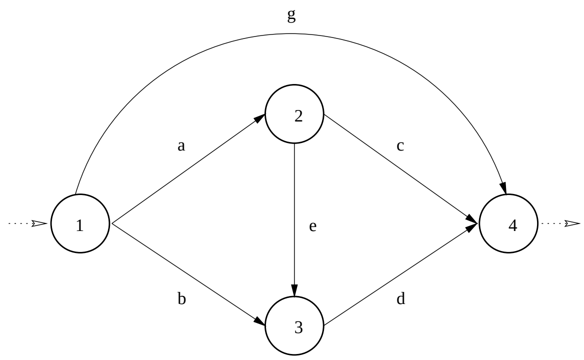
So, we have

$$\sum_{e\in p} l^e(f^e)+q^p=\min_{s'\in\mathcal{P}_k}\left\{\sum_{e\in s'} l^e(f^e)+q^{s'}\right\}. \qquad (23)$$

From equations (22) and (23), we get

$$\sum_{e\in p} l^e(f^e)+q^p=\sum_{e\in s} l^e(f^e)+q^s. \qquad (24)$$

[3] This is because, for such user game, we can construct a new WE from a given WE by interchanging ϵ units of user j_1's flow on path p_1 with ϵ units of user j_2's flow on path p_2 (where p_1 and p_2 belong to the same OD pair and ϵ is less than or equal to the minimum of j_1's flow on path p_1 and j_2's flow on path p_2).



(a)

Fig. 2. A network that violates the monotonicity of flow rates and the monotonicity of link loads.

Substituting equation (24) into equation (21) yields the result

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e\in p} l^e(f^e) - q^p = 0.$$

2) Let $\boldsymbol{f}$ be a WE at the price q. We construct a new flow $\tilde{\boldsymbol{f}}$ in the following way: If $j \notin \overline{\mathcal{J}_k}$ or $p \notin \overline{\mathcal{P}_k}$, set $\tilde{f}^p_{k,j} = 0$. Otherwise, set

$$\tilde{f}^p_{k,j} = \frac{\Gamma_{k,j} f^p}{\sum_{p\in\mathcal{P}_k} f^p}$$

which is > 0 because $j \in \overline{\mathcal{J}_k}$ and $p \in \overline{\mathcal{P}_k}$. Now, since

$$\tilde{f}^p = \sum_{j\in\overline{\mathcal{J}_k}} \tilde{f}^p_{k,j} = f^p, \ \forall\, p \in \mathcal{P}$$

and

$$\tilde{\Gamma}_{k,j} = \sum_{p\in\overline{\mathcal{P}_k}} \tilde{f}^p_{k,j} = \Gamma_{k,j}, \ \forall\, j \in \mathcal{J}_k, k \in \mathcal{W}.$$

$\tilde{\boldsymbol{f}}$ is a WE such that $\tilde{f}^p_{k,j} > 0$ for all $p \in \overline{\mathcal{P}_k}$, $j \in \overline{\mathcal{J}_k}$, and for all k. **Q.E.D.**

It is informative to understand how link loads and users' flow rates change with prices. There are two natural conjectures in this context: As the price of a particular path increases, the amount of data transmitted on the other paths increase. Similarly, the flow rate of each user is a nondecreasing function of the price vector. These results were proven for networks with parallel links in [13]. The same results do not generalize to a general network topology, however.

In a general network where there are no prices and users have fixed demands, improving the latency function of one link (i.e., replacing $l^e(x)$ with $\overline{l^e}(x)$ such that $\overline{l^e}(x) \le l^e(x)\ \forall\ x$ for some link e) while keeping the rest unchanged, may cause all users to encounter higher latency costs. This phenomenon is known as the Braess' Paradox. We next demonstrate such a counterintuitive phenomenon in a network with users with elastic utilities. Consider the example in Figure 2, where a single user sends flow from node 1 to node 4. Assume that the user's utility function is $u(\Gamma) = 184\sqrt{8}\Gamma^{0.5}$, and the link latency functions are given by

$$l^a(f^a) = 10f^a,\ l^b(f^b) = f^b,\ l^c(f^c) = f^c,$$

$$l^d(f^d) = 10f^d,\ l^e(f^e) = f^e,\ l^g(f^g) = f^g.$$

Given the price vector q, where $q^{\{a,c\}} = 50$, $q^{\{b,d\}} = 50$, $q^{\{a,e,d\}} = 10$, $q^{\{g\}} = 90$, the path flows at the WE are $f^{\{a,c\}} = f^{\{b,d\}} = f^{\{a,e,d\}} = f^{\{g\}} = 2$. Consider another price vector $\overline{q}$ where we increase the price of path $\{a, e, d\}$ to 14. Given $\overline{q}$, the path flows at the new WE are $\overline{f}^{\{a,c\}} = \overline{f}^{\{b,d\}} \approx 3.032$, $\overline{f}^{\{a,e,d\}} \approx 0.792$, $\overline{f}^{\{g\}} \approx 1.2721$. However,

$$\overline{f}^{\{g\}} < f^{\{g\}} = 2,\ \text{and}\ \overline{\Gamma} \approx 8.1281 > \Gamma = 8.$$

This shows that at a higher price vector, the flow on an alternative path decreases and the total flow rate of the user increases. We will study Braess' paradox in general networks in more detail in Sec. V.

B. Monopoly Price, Social Optimum, and Performance

In this section, we provide an explicit characterization of the monopoly price and compare the system performance at the monopoly equilibrium with the social optimum. Recall that the monopoly problem is

$$\max_{\boldsymbol{q}\ge 0} \sum_p q^p f^p(\boldsymbol{q}),$$

where $f^p(\boldsymbol{q})$ is the flow on path p at a WE for a given price vector $\boldsymbol{q}$. Under Assumptions 1 and 2, we can assume

$$0 \le q^p \le \min_{j\in\mathcal{J}_k} u'_{k,j}(0),\ \forall\ p \in \mathcal{P}_k,\ k,$$

and by an argument similar to the one given in [13], we can show that $f^p(\boldsymbol{q})$ is continuous in $\boldsymbol{q}$ for all p. Therefore, problem (9) has an optimal solution. We now look at the following proposition which is essential to our analysis.

Proposition 3: Let Assumptions (1) and (2) hold, and let $(\boldsymbol{q}, \boldsymbol{f})$ be an ME. Let $\overline{\mathcal{P}} = \cup_k \overline{\mathcal{P}_k}$ where $\overline{\mathcal{P}_k} = \{p | p \in \mathcal{P}_k, f^p > 0\}$. Then $q^p > 0,\ \ \forall\ p \in \overline{\mathcal{P}}$.

Proof: To arrive at a contradiction, we assume that there exists a path $p' \in \overline{\mathcal{P}_{k'}}$ and $q^{p'} = 0$. From Lemma 3, we know that since $p' \in \overline{\mathcal{P}_k}$, there exists some WE such that for all $j \in \overline{\mathcal{J}_{k'}}$, $f^{p'}_{k',j} > 0$. Since $u_{k',j}$ is strictly concave and $u'_{k',j}$ is continuous, we can pick an $\epsilon > 0$ such that for every $j \in \overline{\mathcal{J}_{k'}}$, there exist a $0 < \delta_j < f^{p'}_{k',j}$ satisfies the following equation.

$$u'_{k',j}(\Gamma_{k',j} - \delta_j) - u'_{k',j}(\Gamma_{k',j}) = \epsilon$$

Notice that for every $e \in p'$, $f^e > \sum_{j\in\overline{\mathcal{J}_{k'}}} \delta_j$. We define a new price vector $\overline{\boldsymbol{q}}$ as

$$\overline{q}^p = q^p + \sum_{e|e\in p, e\in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j\in\overline{\mathcal{J}_{k'}}} \delta_j) \right) + \epsilon, \quad \text{if } p \in \mathcal{P}_{k'};$$

$$\overline{q}^p = q^p + \sum_{e|e\in p, e\in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j\in\overline{\mathcal{J}_{k'}}} \delta_j) \right), \quad \text{otherwise.} \tag{25}$$

Since l is strictly increasing, $\overline{q}^p > q^p$ if $p \in \mathcal{P}_{k'}$ and $\overline{q}^p \geq q^p$ otherwise. Now consider the flow $\overline{\boldsymbol{f}}$ that satisfies the following conditions:

$$\begin{aligned} \overline{f}^{p'}_{k',j} &= f^{p'}_{k',j} - \delta_j, \quad \forall\ j \in \overline{\mathcal{J}_{k'}} \\ \overline{f}^{p}_{k,j} &= f^{p}_{k,j}, \quad \text{otherwise.} \end{aligned} \tag{26}$$

Now, we will show $\overline{\boldsymbol{f}}$ is a WE for the price vector $\overline{\boldsymbol{q}}$. From Eqs. (26), we have

$$\begin{aligned} \overline{f}^e &= f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j,\ \text{if } e \in p'; \\ \overline{f}^e &= f^e,\ \text{otherwise} \\ \overline{\Gamma}_{k',j} &= \Gamma_{k',j} - \delta_j,\ \forall\ j \in \overline{\mathcal{J}_{k'}}; \\ \overline{\Gamma}_{k,j} &= \Gamma_{k,j},\ \text{otherwise .} \end{aligned} \tag{27}$$

From Eqs. (25) and (27), we see that for every $p \in \mathcal{P}_{k'}$ and $j \in \overline{\mathcal{J}}_{k'}$

$$\begin{aligned} & u'_{k',j}(\overline{\Gamma}_{k',j}) \\ = \ & u'_{k',j}(\Gamma_{k',j} - \delta_j) \\ = \ & u'_{k',j}(\Gamma_{k',j}) + \epsilon \\ \leq \ & \sum_{e \in p} l^e(f^e) + q^p + \epsilon \\ = \ & \sum_{e \in p} l^e(f^e) - \\ & \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j) \right) + \overline{q}^p \\ = \ & \sum_{e \in p} l^e(f^e) - \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(\overline{f}^e) \right) + \overline{q}^p \\ = \ & \sum_{e \in p} l^e(\overline{f}^e) + \overline{q}^p. \end{aligned} \tag{28}$$

We know $\overline{f}^p_{k',j} > 0$ iff $f^p_{k',j} > 0$. Hence, the equality of Eq. (28) holds if $\overline{f}^p_{k',j} > 0$. Similarly, for every $p \in \mathcal{P}_k$, $j \in \overline{\mathcal{J}}_k$, and $k \neq k'$

$$\begin{aligned} & u'_{k,j}(\overline{\Gamma}_{k,j}) \\ = \ & u'_{k,j}(\Gamma_{k,j}) \\ \leq \ & \sum_{e \in p} l^e(f^e) + q^p \\ = \ & \sum_{e \in p} l^e(f^e) - \\ & \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j) \right) + \overline{q}^p \\ = \ & \sum_{e \in p} l^e(\overline{f}^e) + \overline{q}^p. \end{aligned} \tag{29}$$

Again, the equality of Eq. (29) holds if $\overline{f}^p_{k,j} > 0$. For price $\overline{\boldsymbol{q}}$, Eqs. (28) and (29) show that $\overline{\boldsymbol{f}}$ satisfies the first order necessary and sufficient conditions:

$$u'_{k',j}(\overline{\Gamma}_{k',j}) - \sum_{e \in p} l^e(\overline{f}^e) - \overline{q}^p \begin{cases} = 0, & \text{if } \overline{f}^p_{k,j} > 0; \\ \leq 0, & \text{if } \overline{f}^p_{k,j} = 0. \end{cases}$$

Therefore, $\overline{\boldsymbol{f}}$ is a WE with price $\overline{\boldsymbol{q}}$. However, since $\overline{q}^{p'} > q$ and $\overline{f}^{p'} > 0$

$$\overline{f}^{p'} \overline{q}^{p'} > 0. \tag{30}$$

Then equation (30) together with $\overline{\boldsymbol{q}} \geq \boldsymbol{q}$ and $q^{p'} = 0$

$$\sum_{p \in \mathcal{P}} \overline{f}^p \overline{q}^p = \overline{f}^{p'} \overline{q}^{p'} + \sum_{p \neq p'} \overline{f}^p \overline{q}^p > \sum_{p \neq p'} f^p q^p = \sum_{p \in \mathcal{P}} f^p q^p.$$

Therefore, $(\boldsymbol{f}, \boldsymbol{q})$ is not an ME and this yields a contradiction. Hence, the result follows. **Q.E.D.**

Now, we can derive an explicit characterization of the monopoly prices. Let $(\boldsymbol{f}, \boldsymbol{q})$ be a ME and for each k, let $\mathcal{I}_k = \{1, \cdots, |\mathcal{P}_k|\}$ be the set of indices of $\mathcal{P}_k$ and $\{1, \cdots, |\mathcal{J}_k|\}$ be the set of indices of $\mathcal{J}_k$. We also denote p^i_k to be the i^{th} path for OD pair k. Without loss of generality, we assume that user $1 \in \overline{\mathcal{J}_k}$ and path $1 \in \overline{\mathcal{P}}_k$ for every k such that $\mathcal{J}_k \neq \emptyset$. Using the necessary and sufficient optimality conditions of a WE at a price vector $\boldsymbol{q}$ together with Lemma 3, we can see that if $(\boldsymbol{f}, \boldsymbol{q})$ is a ME, then $([f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}}, \boldsymbol{q})$ is an optimal solution of the following problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{p \in \overline{\mathcal{P}}} q^p f^p & (31) \\ \text{subject to} \quad & u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p^i_k} l^e(\sum_{p | e \in p} f^p) - q^{p^i_k} = 0, & \\ & \qquad \forall\ p^i_k \in \overline{\mathcal{P}_k}, k \in \mathcal{W} & (32) \\ & u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p^i_k} l^e(\sum_{p | e \in p} f^p) - q^{p^i_k} \leq 0, & \\ & \qquad \forall\ p^i_k \notin \overline{\mathcal{P}_k}, k \in \mathcal{W} & (33) \\ & u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p^1_k} l^e(\sum_{p | e \in p} f^p) - q^{p^1_k} = 0, & \\ & \qquad \forall\ j \in \overline{\mathcal{J}_k} - \{1\}, k \in \mathcal{W} & (34) \\ & u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p^1_k} l^e(\sum_{p | e \in p} f^p) - q^{p^1_k} \leq 0, & \\ & \qquad \forall\ j \notin \overline{\mathcal{J}_k}, \forall\ k \in \mathcal{W} & (35) \\ & \sum_{p \in \overline{\mathcal{P}_k}} f^p = \sum_{j \in \overline{\mathcal{J}_k}} \Gamma_{k,j}, \forall\ k \in \mathcal{W} & (36) \\ & \Gamma_{k,j} \geq 0,\ \forall\ j \in \overline{\mathcal{J}_k}, k \in \mathcal{W}, & \\ & f^p \geq 0,\ \forall\ p \in \mathcal{P}, & \\ & q^p \geq 0,\ \forall\ p \in \mathcal{P}. & \end{aligned}$$

Note that we use the necessary and sufficient optimality conditions for a WE to write problem (31) in variables $([f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}}, \boldsymbol{q})$ instead of variables $(\boldsymbol{f}, \boldsymbol{q})$ and use Lemma 3 to eliminate the redundant constraints. This reduction in the dimension of the feasible set allows us to show that the regularity constraint qualification is satisfied (i.e., the constraint gradients of problem (31) are linearly independent at the optimal solution). Thus, the nonconvex problem (31) admits Lagrange multipliers, which will be the key in proving the subsequent proposition. This is stated in the following Lemma. The proof can be found in Appendix A.

Lemma 4: The constraint gradients of problem (31) at any feasible solution $([f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}}, \boldsymbol{q})$ *are linearly independent.*

Proposition 4: Let Assumptions (1) and (2) hold. Assume further that $u_{k,j}$ is twice continuously differentiable for each j and k, and l^e is continuously differentiable for each e. Let $(\boldsymbol{f}, \boldsymbol{q})$ be an ME, then for every path p in $\overline{\mathcal{P}_k}$, we have

$$q^p = \left(\sum_{e \in p} f^e (l^e)'(f^e)\right) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{-\sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}}. \quad (37)$$

Proof: Let $([f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}}, \boldsymbol{q})$ be an optimal solution of problem (31). Define $\mathcal{I}_k$ to be the set of the indices of the paths in $\mathcal{P}_k$ and $\overline{\mathcal{I}}_k$ to be the set of indices of the paths in $\overline{\mathcal{P}}_k$. By Lemma 4, there exist Lagrange Multipliers for problem (31). We assign λ_k^i to the constraints (32) and (33), $\mu_{k,j}$ to the constraints (34) and (36), and finally $\xi_{k,j}$ to the constraints (35). The Lagrangian function $L(\boldsymbol{q}, \boldsymbol{f}, \lambda, \mu, \xi)$ can be written as

$$\begin{aligned} L(q, f, \lambda, \mu, \xi) = \\ \sum_{p \in \overline{\mathcal{P}}} q^p f^p + \sum_k \sum_{i \in \mathcal{I}_k} \lambda_k^i [u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p_k^i} l^e(f^e) - q^{p_k^i}] \\ + \sum_k \sum_{j \in \overline{\mathcal{J}_k} - \{1\}} \mu_{k,j} [u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e(f^e) - q^{p_k^1}] \\ + \sum_k \sum_{j \notin \overline{\mathcal{J}_k}} \xi_{k,j} [u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e(f^e) - q^{p_k^1}] \\ + \sum_k \mu_{k,1} [\sum_{p \in \overline{\mathcal{P}_k}} f^p - \sum_{j \in \overline{\mathcal{J}_k}} \Gamma_{k,j}]. \end{aligned}$$

If the monopoly price vector $\boldsymbol{q}$ is not greater than $\mathbf{0}$, we can find another monopoly price vector $\boldsymbol{q}'$ such that

$$(q')^p = q^p, \text{ if } p \in \overline{\mathcal{P}}; \text{ but } (q')^p > 0, \ \forall\ p.$$

Therefore, without loss of generality we can assume that the ME price vector $\boldsymbol{q}$ satisfies $\boldsymbol{q} > \mathbf{0}$. So, for each OD pair k,

$$\frac{\partial L}{\partial q^{p_k^1}} = 0 \rightarrow f^{p_k^1} = \lambda_k^1 + \sum_{j \in \overline{\mathcal{J}_k} - \{1\}} \mu_{k,j} + \sum_{j \notin \overline{\mathcal{J}_k}} \xi_{k,j}, \quad (38)$$

$$\frac{\partial L}{\partial q^{p_k^i}} = 0 \rightarrow f^{p_k^i} = \lambda_k^i, \ \forall\ i \in \overline{\mathcal{I}_k}, \ i \neq 1, \quad (39)$$

$$\frac{\partial L}{\partial q^{p_k^i}} = 0 \rightarrow 0 = \lambda_k^i, \ if\ i \notin \overline{\mathcal{I}_k}. \quad (40)$$

Recall that $\overline{\mathcal{P}} = \{p \mid p \in \mathcal{P}, f^p > 0\}$ and problem (31) is defined on $\overline{\mathcal{P}}$ but not $\mathcal{P}$. Therefore, for each $f^p \in \overline{\mathcal{P}}$, we have

$$\begin{aligned} \frac{\partial L}{\partial f^{p_k^i}} = 0 \rightarrow \\ q^{p_k^i} - \sum_m \left[\sum_{n \in \mathcal{I}_m} \lambda_m^n \left(\sum_{e | e \in p_k^i, e \in p_m^n} (l^e)'(f^e)\right)\right] - \\ \sum_m \left(\sum_{j \in \overline{\mathcal{J}_m} - \{1\}} \mu_{m,j} + \sum_{j \notin \overline{\mathcal{J}_m}} \xi_{m,j}\right) \left(\sum_{e | e \in p_k^i, e \in p_m^1} (l^e)'(f^e)\right) \\ + \mu_{k,1} = 0. \end{aligned}$$

Simplifying the preceding equation, we get

$$\begin{aligned} q^{p_k^i} - \sum_m \Bigg[\sum_{n \in \overline{\mathcal{I}_m} - 1} \lambda_m^n \left(\sum_{e | e \in p_k^i, e \in p_m^n} (l^e)'(f^e)\right) + \\ \left(\lambda_m^1 + \sum_{j \in \overline{\mathcal{J}_m} - \{1\}} \mu_{m,j} + \sum_{j \notin \overline{\mathcal{J}_m}} \xi_{m,j}\right) \sum_{e | e \in p_k^i, e \imath p_m^1} (l^e)'(f^e)\Bigg] \\ + \mu_{k,1} = 0. \end{aligned}$$

Substituting Eqs. (40) into the equation above, we have

$$\begin{aligned} q^{p_k^i} - \sum_m \Bigg[\sum_{n \in \overline{\mathcal{I}_m} - 1} f^{p_m^n} \left(\sum_{e | e \in p_k^i, e \in p_m^n} (l^e)'(f^e)\right) + \\ f^{p_m^1} \left(\sum_{e | e \in p_k^i, e \in p_m^1} (l^e)'(f^e)\right)\Bigg] + \mu_{k,1} = 0 \end{aligned}$$

and then

$$\begin{aligned} q^{p_k^i} - \sum_{e \in p_k^i} \left[\left(\sum_m \sum_{\{n | n \in \overline{\mathcal{I}_m}, e \in p_m^n\}} f^{p_m^n}\right) (l^e)'(f^e)\right] \\ + \mu_{k,1} = 0. \end{aligned}$$

Therefore,

$$q^{p_k^i} - \sum_{e \in p_k^i} f^e (l^e)'(f^e) + \mu_{k,1} = 0, \ \forall\ k, i \in \overline{\mathcal{I}_k}. \quad (41)$$

Also for the set of flow rate variables, we have:

$$\frac{\partial L}{\partial \Gamma_{k,1}} = 0 \rightarrow u''_{k,1}(\Gamma_{k,1}) \left(\sum_i \lambda_k^i\right) - \mu_{k,1} = 0, \ \forall\ k, \quad (42)$$

$$\frac{\partial L}{\partial \Gamma_{k,j}} = 0 \rightarrow \mu_{k,j} u''_{k,j}(\Gamma_{k,j}) - \mu_{k,1} = 0, \ \forall\ j \in \overline{\mathcal{J}_k} - 1, k, \quad (43)$$

$$\frac{\partial L}{\partial \Gamma_{k,j}} = 0 \rightarrow \xi_{k,j} u''_{k,j}(\Gamma_{k,j}) \leq 0, \ \forall\ j \notin \overline{\mathcal{J}_k}, \forall\ k. \quad (44)$$

Since $\xi_{k,j} \leq 0$ for all k and j, and $u''_{k,j}(\Gamma_{k,j}) \leq 0$ for all k and j, Eq. (44) implies that $\xi_{k,j} = 0$ for all k and j. Therefore, summing all the equations in (43), we get

$$\sum_{j \in \overline{\mathcal{J}_k} - 1} \mu_{k,j} = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k} - 1} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall\ k. \quad (45)$$

From Eqs. (42) and (45), we obtain

$$\sum_{j \in \overline{\mathcal{J}_k} - 1} \mu_{k,j} + \left(\sum_i \lambda_k^i\right) = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall\ k. \quad (46)$$

Eqs. (38), (39), (40), and (46) imply that

$$\sum_{p \in \overline{\mathcal{P}_k}} f^p = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall\ k. \quad (47)$$

Substituting Eq. (47) into Eq. (41) we

$$q^p = \left(\sum_{e \in p} f^e (l^e)'(f^e)\right) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{-\sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}}.$$

Q.E.D.

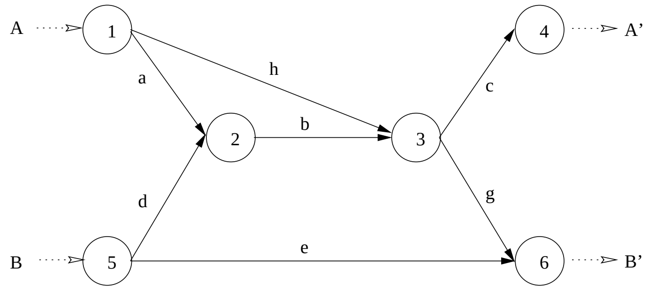


Fig. 3. A simple general network

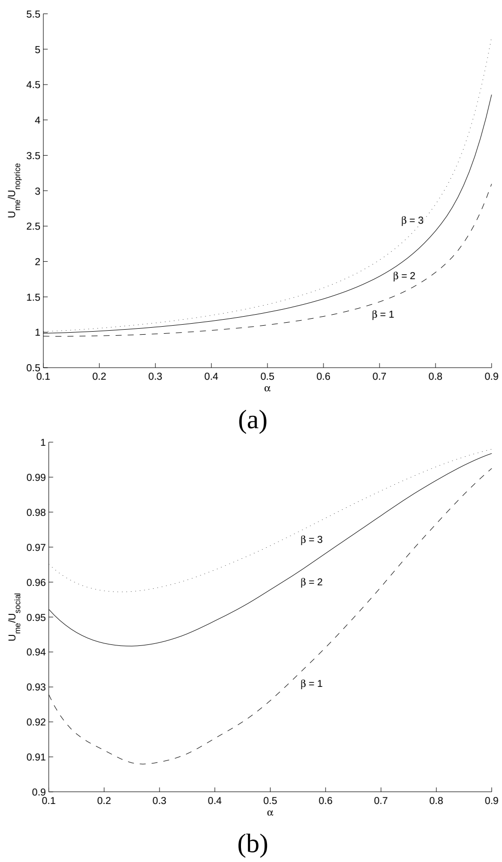


Fig. 4. a) Performance of ME over WE at price 0 b) Performance of ME over Social Optimum

This proposition shows that the monopoly price is given by two terms: The first term is the "marginal congestion cost" (which corresponds to a Pigovian tax on the externality created by the users [11]). This amounts to charging every user the marginal increase in congestion by sending an extra unit of data. It is well-known that this is the price that a network planner maximizing the total system performance would charge in order to force users to internalize the congestion effects (resulting in the social optimum) [12, 13]. The second term is a markup above this given by the profit-maximizing objective of the service provider. Which of these two terms is dominant will determine the relative performance of the monopoly equilibrium compared to a situation without prices and to the social optimum.

Example 1: We consider a simple general network as given in Figure (3). We have two users (A and B) and 4 paths ($\{h,c\},\{a,b,c\},\{d,b,g\},\{e\}$) in the network. The utility functions of the users and the latency functions of the links are given by

$$u_A(\Gamma_A) = 200(\Gamma_A)^{\alpha},\ u_B(\Gamma_B) = 200(\Gamma_B)^{\alpha},$$

$$l^e(f^e) = (f^e)^{\beta},\ \forall\ e \in \mathcal{E}.$$

Let U_{me}, U_{social}, and U_0 be the total system utility,

$$\sum_k \left[\sum_{j\in\mathcal{J}_k} u_{k,j}(\Gamma_{k,j}) - \sum_{p\in\mathcal{P}_k} \left(\sum_{e\in p} l^e(f^e) \right) f^p \right]$$

at the monopoly equilibrium, social optimum, and at the WE at 0 prices, respectively. The plot of the ratios U_{me}/U_0 and U_{me}/U_{social} as a function of different values of α and β are given in Figures 4(a) and (b), respectively.

The results shown in Figure 4 are intuitive. The first panel shows that as β increases, performance of the monopoly equilibrium improves relative to an equilibrium without any prices (e.g., as in [13]). This is because higher values of β imply that latencies are more sensitive to link load and thus correspond to greater congestion effects (externalities), which are internalized in the monopoly equilibrium, but not in the equilibrium without prices. It also shows that performance improves as α increases. Greater α corresponds to a more linear utility function, and as Eq. (37) shows the markup is smaller when the utility function is less concave, reducing the monopoly distortions. The second panel is similar, however, it shows that the performance of the monopoly equilibrium relative to social optimal with respect to α is non-monotonic. The reason why values of α close to 1 improve the performance of the monopoly equilibrium is the same as above. However, the monopoly equilibrium also performs relatively well for very small values of α. This is because, in this case, even though the markup is substantial, individuals have a very high marginal utility of data transmission at low flow rates and choose not to reduce their flow rates much in response to this high markup, thus system performance does not suffer much.

V. BRAESS' PARADOX

Braess' Paradox, first defined by Braess in 1968 [16], states the counterintuitive fact that adding a link to a network might cause all users to be worse off than in the previous equilibrium. This phenomenon is due to the non-cooperative nature of the selfish users, as each user only wants to minimize his travel cost without considering the travel costs of other users. Braess' Paradox has been recognized and studied in different kinds of networks. For example, Hagstrom and Abrams [28] outlined a characterization of Braess' Paradox in traffic networks. Steinberg and Zangwill [32] gave necessary and sufficient conditions for the existence of Braess' Paradox in a transportation network under limited assumptions. Cohen and Kelly [26] also studied an example of Braess' Paradox in a queueing network. A detailed survey of research on Braess' Paradox can be found in [23] and [20].

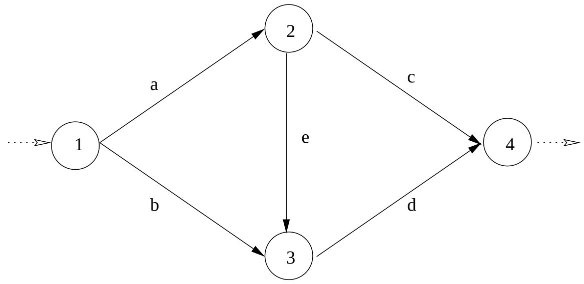


Fig. 5. An example of Braess' Paradox under pricing

The observation of Braess' Paradox motivated research in methods of upgrading the network capacity without degrading network performance. Some proposed methods were:

1) Multiplying the capacity of each link by some constant factor $\alpha > 1$ [29, 30] or a link dependent factor $\alpha_l >$ number of users [30].
2) Adding a direct link between the source and the destination [29, 30, 31].
3) Increasing the capacity of a direct link [30].

These methods emerged as results of studies in sensitivity analysis. In particular, methods (1)-(2) are motivated by the sensitivity result that states that the equilibrium cost of an OD pair is a monotone non-decreasing function of the corresponding demand [22, 30, 33]. Method (3) is motivated by the sensitivity result that states that improving the link latency function on only one link results in a decrease of the latency on that link [22]. The methods proposed above are constrained by assumptions on link latency functions or users. However, whether any assumption has been made or not, we can see that these methods are limited.

Braess' Paradox can be arbitrarily severe in many networks [23]. Most of the network design problems related to Braess' Paradox, such as the ones mentioned above, focus on finding ways to avoid this undesired but common phenomenon. Therefore, in the remainer of this section, we will examine the implications of profit maximizing prices on Braess' Paradox.

Hagstrom and Abrams [28] gave a definition of Braess' Paradox in a network without pricing: A Braess' Paradox occurs if there exists some other distribution of flows for which some flow have improved travel costs and no flow has worse travel cost than in the equilibrium. This is a generalization of the classical Braess' Paradox which refers to change in network performance by adding/deleting a link. In [28], Hagstrom and Abrams showed a network which experiences a generalized Braess' Paradox but no classical Braess' Paradox.

In a network without pricing, at a WE, all flows on the paths that belong to the same OD pair experience the same latency cost. Therefore we can restate the generalized definition of Braess' Paradox given above as: A Braess' Paradox occurs in a network if there exists some other distribution of flows for which *some paths* have improved latency costs and *no path* has a worse latency cost than in the equilibrium. At a WE with prices, flows on different paths may have different latency costs. Therefore, there might exist some other flow distribution for which some paths have improved latency costs and no path has worse latency cost, but some flow which switched from one path to another has worse latency cost than in the equilibrium. Such a situation should not be considered as a Braess' Paradox. For an example, let's consider the network in Figure (5). A single user sends data from node 1 to node 4.

$$u(\Gamma) = 368\sqrt{6}\Gamma^{0.5}$$

The link latency functions and path prices are as follows:

$$l^a(f^a) = (f^a)^2,\ l^b(f^b) = 5f^b,\ l^c(f^c) = 5f^c,$$
$$l^d(f^d) = (f^d)^2,\ l^e(f^e) = 0, q^{\{a,c\}} = 182.5619,$$
$$q^{\{b,d\}} = 182.5619,\ q^{\{a,e,d\}} = 193.5619$$

The path flows at the WE are

$$f^{\{a,c\}} = f^{\{b,d\}} = 2,\ f^{\{a,e,d\}} = 1.$$

The latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 19,\ l^{\{a,e,d\}} = 18.$$

We next consider moving 0.5 units of flow from path $\{a, e, d\}$ to paths $\{a, c\}$ and $\{b, d\}$. The resulting path flows are

$$f^{\{a,c\}} = f^{\{b,d\}} = 2.5,\ f^{\{a,e,d\}} = 0,$$

and the corresponding latency costs are

$$l^{\{a,c\}} = l^{\{b,d\}} = 18.75,\ l^{\{a,e,d\}} = 12.5.$$

Hence, the flow that is moved from $\{a, e, d\}$ to alternative paths experiences a higher latency cost. It can be seen that there is no flow distribution in which all flows experience improved latency costs.

We next give two alternative definitions of Braess' Paradox under pricing. The following notation will be useful in the definitions. Consider two feasible flow distributions $\boldsymbol{f}$ and $\overline{\boldsymbol{f}}$ such that

$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall\ k, j.$$

Let $\boldsymbol{h}$ be the path flow vector defined in (3) and Δ be a transformation matrix such that

$$\Delta \cdot \boldsymbol{h} = \overline{\boldsymbol{h}}. \tag{48}$$

Hence, $\Delta_{i,j} f^j$ represents the amount of flow that is moved from path j to path i. Note that there are infinitely many transformation matrices Δ satisfying Eq. (48).

Definition 3: **(Strong Braess' Paradox):** *Let G be a general network. Given a price $\boldsymbol{q}$, let $\boldsymbol{f}$ be a WE. Let $l^p(\boldsymbol{h})$ be the latency cost of routing one unit of flow on path p as defined in Eq. (7). A Strong Braess's Paradox occurs if there exists some other distribution of path flows, $\overline{\boldsymbol{h}}$, and a transformation Δ such that*

$$\Delta \cdot \boldsymbol{h} = \overline{\boldsymbol{h}}$$
$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall\ k, j$$
$$l^p(\boldsymbol{h}) \geq l^{p'}(\overline{\boldsymbol{h}}),\quad \textit{for all } p, p' \textit{ with } \Delta_{p',p} \neq 0, \tag{49}$$

with strict inequality for some p, p', where $\Delta_{p',p}$ is the (p', p) entry of matrix Δ.

Under condition (49), no flow experiences a higher latency cost than in the WE. For a price vector in which the prices of all the paths that belong to an OD pair are the same, Definition (3) is consistent with the definition of Braess' Paradox in a network without pricing. For an example of Strong Braess' Paradox, we can consider the same network shown in Figure (5). The user sends data from node 1 to node 4. The user's utility function, link latency functions, and path prices are given as:

$$\Gamma = 184\sqrt{6}\Gamma^{0.5},\ l^a(f^a) = 10f^a,\ l^b(f^b) = f^b,$$

$$l^c(f^c) = f^c,\ l^d(f^d) = 10f^d,\ l^e(f^e) = f^e,$$

$$q^{\{a,c\}} = 50,\ q^{\{b,d\}} = 50,\ q^{\{a,e,d\}} = 10.$$

The path flows at the WE are

$$f^{\{a,c\}} = f^{\{b,d\}} = f^{\{a,e,d\}} = 2.$$

The latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 42;\ l^{\{a,e,d\}} = 82.$$

We move one unit of flow from path $\{a,e,d\}$ to each of paths $\{a,c\}$ and $\{b,d\}$ in order to get a new flow distribution: $f^{\{a,c\}} = f^{\{b,d\}} = 3$ and $f^{\{a,e,d\}} = 0$. In this flow distribution, the latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 33.$$

Each unit of flow experiences a latency cost equal to 33, which is less than the latency cost at the WE. Note that this price vector is not a monopoly price vector. Later, we will show that under monopoly prices, Strong Braess' Paradox does not occur.

Conditions in Definition (3) state that when Strong Braess' Paradox occur, at the new flow distribution, some flows have lower latency cost and no flow has a higher latency cost. We will next relax these conditions so that some flow may encounter higher latency costs at the new flow distribution, but on average the latency encountered by the total flow will decrease. This leads to the following definition.

Definition 4: **(Weak Braess' Paradox):** *Let G be a general network. Given a price q, let f be a WE. Let $l^p(\boldsymbol{h})$ be the latency cost of routing one unit of flow on path p under a path flow $\boldsymbol{h}$ and $\boldsymbol{l}(\boldsymbol{h}) = [l^p(\boldsymbol{h})]_{p\in\mathcal{P}}$ be the path latency vector. A Weak Braess's Paradox occurs if there exists some other distribution of path flows, $\overline{\boldsymbol{h}}$, under price q and a transformation Δ such that*

$$\Delta \cdot \boldsymbol{h} = \overline{\boldsymbol{h}}$$

$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall\ k, j \tag{50}$$

for some p'

$$l^p(\boldsymbol{h}) \geq \Delta'_p \cdot \boldsymbol{l}(\overline{\boldsymbol{h}}), \forall\ p, \tag{51}$$

with strict inequality for some p', where Δ_p is the pth column of Δ.

Condition (49) in Definition (3) imply Condition (51) in Definition (4). Therefore, if Strong Braess' Paradox occurs, then Weak Braess' Paradox also occurs. The following example shows that the reverse implication is not true. Consider the network in Figure (5) with different functions:

$$\Gamma = 368\sqrt{6}\Gamma^{0.5},\ l^a(f^a) = (f^a)^2,\ l^b(f^b) = 3f^b,$$

$$l^c(f^c) = 5f^c,\ l^d(f^d) = (f^d)^2,\ l^e(f^e) = 0,$$

$$q^{\{a,c\}} = 193.4649,\ q^{\{b,d\}} = 201.7149,\ q^{\{a,e,d\}} = 197.2149$$

The path flows at the WE are $f^{\{a,c\}} = 2$, $f^{\{b,d\}} = 1.5$, $f^{\{a,e,d\}} = 1$ and the path latency costs are:

$$l^{\{a,c\}} = 19;\ l^{\{b,d\}} = 10.75;\ l^{\{a,e,d\}} = 15.25.$$

Next, we move 0.5 units of flow from path $\{a,e,d\}$ to each of the paths $\{a,c\}$ and $\{b,d\}$ to get a new flow distribution: $f^{\{a,c\}} = 2.5$, $f^{\{b,d\}} = 2$ and $f^{\{a,e,d\}} = 0$. In this flow distribution, the latency costs of the paths are

$$l^{\{a,c\}} = 18.75;\ l^{\{b,d\}} = 10;\ l^{\{a,e,d\}} = 10.25.$$

We see that

$$18.75 < 19;\ 10 < 10.75$$

$$0.5 \times 18.75 + 0.5 \times 10 = 14.375 < 15.25.$$

Therefore, in this example, Weak Braess' Paradox occurs. However, it can be seen that there exists no flow distribution in which all flows will encounter lower latencies than at the WE. Therefore, Strong Braess' Paradox does not occur.

We next show that under monopoly prices, Weak Braess' Paradox does not occur, which also implies that under monopoly prices, there can be no Strong Braess' Paradox.

Proposition 5: Weak Braess' Paradox does not occur under monopoly prices.

Proof: We consider a general network G. Let $(\boldsymbol{f}, \boldsymbol{q})$ be an ME. Suppose that Weak Braess' Paradox occurs under the monopoly price $\boldsymbol{q}$. Then there exists another flow distribution $\overline{\boldsymbol{f}}$ satisfying Conditions (50) - (51). Now let us consider the price vector $\overline{\boldsymbol{q}}$ defined by

$$\overline{q}^p = u'_{k,j'}(\overline{\Gamma}_{k,j'}) - l^p(\overline{\boldsymbol{h}}) \text{ for some } j' \in \mathcal{J}_k,\ \text{if } p \in \overline{\mathcal{P}}$$

$$\overline{q}^p = \infty,\ \text{if } p \notin \overline{\mathcal{P}}.$$

It can be seen that $\overline{\boldsymbol{f}}$ is a WE at price $\overline{\boldsymbol{q}}$. In the following, we will examine the profit that the service provider makes under price $\bar{\boldsymbol{q}}$.

$$\begin{aligned}
& \sum_p \bar{f}^p \bar{q}^p \\
= & \sum_k \sum_{p\in\mathcal{P}_k} \bar{f}^p \bar{q}^p \\
= & \sum_k \sum_{p\in\mathcal{P}_k} \bar{f}^p \left(u'_{k,j'}(\bar{\Gamma}_{k,j'}) - l^p(\bar{\boldsymbol{h}})\right) \\
= & \left(\sum_k u'_{k,j'}(\bar{\Gamma}_{k,j'}) \sum_{p\in\mathcal{P}_k} \bar{f}^p\right) - \sum_k \sum_{p\in\mathcal{P}_k} \bar{f}^p l^p(\bar{\boldsymbol{h}}) \\
= & \left(\sum_k u'_{k,j'}(\bar{\Gamma}_{k,j'}) \sum_{j\in\mathcal{J}_k} \bar{\Gamma}_{k,j}\right) \\
& -\sum_k \sum_{p\in\mathcal{P}_k} \left[\left(\sum_{p'\in\mathcal{P}_k} \Delta_{p,p'} f^{p'}\right) l^p(\bar{\boldsymbol{h}})\right] \\
= & \left(\sum_k u'_{k,j'}(\bar{\Gamma}_{k,j'}) \sum_{j\in\mathcal{J}_k} \bar{\Gamma}_{k,j}\right) \\
& -\sum_k \sum_{p'\in\mathcal{P}_k} \left[f^{p'}\left(\sum_{p\in\mathcal{P}_k} \Delta_{p,p'} l^p(\bar{\boldsymbol{h}})\right)\right] \\
= & \left(\sum_k u'_{k,j'}(\bar{\Gamma}_{k,j'}) \sum_{j\in\mathcal{J}_k} \bar{\Gamma}_{k,j}\right) \\
& -\sum_k \sum_{p'\in\mathcal{P}_k} \left(f^{p'} \Delta'_{p'} \cdot \boldsymbol{l}(\bar{\boldsymbol{h}})\right) \\
> & \left(\sum_k u'_{k,j'}(\Gamma_{k,j'}) \sum_{j\in\mathcal{J}_k} \Gamma_{k,j}\right) \\
& -\sum_k \sum_{p'\in\mathcal{P}_k} \left(f^{p'} l^{p'}(\boldsymbol{h})\right) \\
= & \left(\sum_k u'_{k,j'}(\Gamma_{k,j'}) \sum_{p'\in\mathcal{P}_k} f^{p'}\right) - \sum_k \sum_{p'\in\mathcal{P}_k} f^{p'} l^{p'}(\boldsymbol{h}) \\
= & \sum_k \sum_{p'\in\mathcal{P}_k} f^{p'} \left(u'_{k,j'}(\Gamma_{k,j'}) - l^{p'}(\boldsymbol{h})\right) \\
= & \sum_{p'} f^{p'} q^{p'}
\end{aligned}$$

The inequality follows from Eqs. (50)-(51). The service provider can make more profit by setting price $\bar{\boldsymbol{q}}$ than $\boldsymbol{q}$. Therefore, $(\boldsymbol{f}, \boldsymbol{q})$ is not an ME, which is a contradiction and shows that Braess' Paradox does not occur under monopoly prices. **Q.E.D.**

APPENDIX I
PROOF OF LEMMA (4)

Proof: Let $([f^p]_{p\in\overline{\mathcal{P}}}, [\Gamma_j]_{j\in\mathcal{J}}, \boldsymbol{q})$ be a feasible solution of problem (31). Let $\mathcal{C}_k$ be the set of constraints gradients for OD pair k at $([f^p]_{p\in\overline{\mathcal{P}}}, [\Gamma_j]_{j\in\mathcal{J}}, \boldsymbol{q})$, and $\mathcal{C} = \cup_k \mathcal{C}_k$. We first show that the vectors in $\mathcal{C}_k$ are linearly independent. Consider the matrix $\boldsymbol{R}$, where each row corresponds to a vector in $\mathcal{C}_k$ (for simplification, we do not include the entries that are 0 in all rows).

$$\begin{pmatrix}
u''_{k,1} & \cdots & 0 & \boldsymbol{S}_{1\times|\mathcal{P}|} & -1 & \cdots & 0 \\
u''_{k,1} & \cdots & 0 & & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \boldsymbol{M}_{(|\mathcal{P}_k|-1)\times|\mathcal{P}|} & \vdots & \vdots & \vdots \\
u''_{k,1} & \cdots & 0 & & 0 & \cdots & -1 \\
 & \cdots & 0 & & -1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \boldsymbol{N}_{|\mathcal{J}_k|\times|\mathcal{P}|} & \vdots & \vdots & \vdots \\
0 & \cdots & u''_{k,|\mathcal{J}_k|} & & -1 & \cdots & 0 \\
-1 & \cdots & -1 & \boldsymbol{T}_{1\times|\mathcal{P}|} & 0 & \cdots & 0
\end{pmatrix}$$

Let $\boldsymbol{r}_i$ be the i^{th} row vector of $\boldsymbol{R}$ and $\boldsymbol{r}_i(x)$ be the entry in $\boldsymbol{r}_i$ corresponding to variable x. Note that $\boldsymbol{M}$ can be a arbitrary matrix, but $\boldsymbol{N}$ is a matrix with each of its row equal to vector $\boldsymbol{S}$.

We first show that the vectors $\{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent. Let $\boldsymbol{r}_i \in \{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$, then there exists x such that $\boldsymbol{r}_i(x) \neq 0$ but $\boldsymbol{r}_j \neq \boldsymbol{0}$ for all $j \neq i$. Therefore, $\{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent. Suppose that $\boldsymbol{r}_1$ can be written as a linear combination of vectors in $\{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. Again, we let $\boldsymbol{r}_i \in \{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$, then there exists x such that $\boldsymbol{r}_i(x) \neq 0$ but $\boldsymbol{r}_j(x) = 0$ for all $j \neq i$. Therefore, $\boldsymbol{r}_1 = \boldsymbol{0}$. However, $\boldsymbol{r}_1 \neq \boldsymbol{0}$ and therefore it cannot be written as a linear combination of vectors in $\{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. As a result, vectors $\{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent.

We then consider the last row, $\boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}$, of $\boldsymbol{R}$. We assume it can be written as a linear combination of the vectors in $\{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$:

$$\boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1} = y_1\boldsymbol{r}_1 + \sum_{i=2}^{|\mathcal{P}_k|} y_i \boldsymbol{r}_i + \sum_{j=1}^{|\mathcal{J}_k|} t_j \boldsymbol{r}_{|\mathcal{P}_k|+j}.$$

For each $\boldsymbol{r}_{i'} \in \{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|}\}$, $\exists\ x$ such that $\boldsymbol{r}_{i'}(x) \neq 0$ but $\boldsymbol{r}_{j'}(x) = 0$ for all $i' \neq j'$. Therefore, y_i is 0 for all $i = 2, \cdots, |\mathcal{P}_k|$ and

$$\boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1} = y_1\boldsymbol{r}_1 + \sum_{j=1}^{|\mathcal{J}_k|} t_j \boldsymbol{r}_{|\mathcal{P}_k|+j}.$$

We also see that

$$\boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}(q^{p_k^1}) = 0, \text{ and}$$

$$\boldsymbol{r}_{k,i}(q^{p_k^1}) = -1, \text{ if } i = 1, |\mathcal{P}_k|+1, \cdots, |\mathcal{P}_k|+|\mathcal{J}_k|.$$

Therefore,

$$y_1 + \sum_{j=1}^{|\mathcal{J}_k|} t_j = 0 \tag{52}$$

However, since each row of $\boldsymbol{N}$ is identical to $\boldsymbol{S}$, Eq. (52) implies all entries in $\boldsymbol{T}$ are $\boldsymbol{0}$. This yields a contradiction. Therefore, $\boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}$ can not be written as a linearly combination of the vectors in $\{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. As a result, the vectors in $\mathcal{C}_k$ are linearly independent.

We next show that the vectors in $\mathcal{C}$ are linearly independent. Let $\mathcal{W}_k$ be the subspace spanned by the vectors in $\mathcal{C}_k$. We will show $\mathcal{W}_k \cap \mathcal{W}_{k'} = \{0\}$ for $k \neq k'$. Assume there exist a vector $w \in \mathcal{W}_k, \mathcal{W}_{k'}$, and $\boldsymbol{w} \neq \mathbf{0}$. Since $\mathcal{J}_k \cap \mathcal{J}_k' = \emptyset$ and $\mathcal{P}_k \cap \mathcal{P}_k' = \emptyset$ if $k \neq k'$. Therefore,

$$\boldsymbol{w}(u''_{k,j}) = 0, \forall\, j; \quad \boldsymbol{w}(u''_{k,j'}) = 0, \forall\, j'; \quad \boldsymbol{w}(q^p) = 0, \forall\, p. \quad (53)$$

Now, we write $\boldsymbol{w}$ as a linear combination of the gradients of $\mathcal{C}_k$:

$$\boldsymbol{w} = y_1' \boldsymbol{r}_1 + \sum_{i=2}^{|\mathcal{P}_k|} y_i' \boldsymbol{r}_i + \left(\sum_{j=1}^{|\mathcal{J}_k|} t_j' \boldsymbol{r}_{|\mathcal{P}_k|+j} \right) + t_0' \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}.$$

For each $\boldsymbol{r}_{i'} \in \{\boldsymbol{r}_2, \cdots, \boldsymbol{r}_{|\mathcal{P}_k|}\}$, $\exists\, x$ such that $\boldsymbol{r}_{i'}(x) \neq 0$ but $\boldsymbol{r}_{j'}(x) = 0$ for all $i' \neq j'$. Therefore, y_i' is 0 for all $i = 2, \cdots, |\mathcal{P}_k|$. Therefore,

$$\boldsymbol{w} = y_1' \boldsymbol{r}_1 + \left(\sum_{j=1}^{|\mathcal{J}_k|} t_j' \boldsymbol{r}_{|\mathcal{P}_k|+j} \right) + t_0' \boldsymbol{r}_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}.$$

Also, since

$$\boldsymbol{w}(q^{p_k^1}) = \boldsymbol{r}_{k,|\mathcal{P}_k|+|\mathcal{J}_k|+1}(q^{p_k^1}) = 0;$$

$$\boldsymbol{r}_{k,i}(q^{p_k^1}) = -1, \text{ for } i = 1, |\mathcal{P}_k|+1, \cdots, |\mathcal{P}_k| + |\mathcal{J}_k|$$

then,

$$y_1' + \sum_{j=1}^{|\mathcal{J}_k|} t_j' = 0 \quad (54)$$

From (53), (54) and the fact that each row of $\boldsymbol{N}$ is identical to $\boldsymbol{S}$, we can see

$$w = [0, \cdots, 0, \boldsymbol{T}, 0, \cdots, 0]$$

By applying the same argument to OD pair k', we also have

$$Sw = [0, \cdots, 0, \boldsymbol{T}', 0, \cdots, 0]$$

However, since $k \neq k'$, $\boldsymbol{T} \neq \boldsymbol{T}'$. So by contradiction, $\mathcal{W}_k \cap \mathcal{W}_{k'} = \{0\}$ for $k \neq k'$. Since the vectors in $\mathcal{C}_k$ form a basis of $\mathcal{W}_k$, we conclude that the vectors in $\mathcal{C}$ are linearly independent. **Q.E.D.**

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