Bayesian Learning in Social Networks

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Motivation

- **Objective:** understand information aggregation in social networks.

- **Model:**
  - Dynamic game with unknown *state of the world*
  - Sequential decisions based on private signals and observation of past actions
  - Payoff conditional on underlying state (same for all agents)

- **Question:** Under what conditions do individuals make correct decisions (or learn the state) as the social network grows bigger?
A Simple Motivating Model

- Model for Bayesian learning on a line [Bikchandani, Hirschleifer, Welch (92), Banerjee (92)]

- Two possible states of the world $\theta \in \{0, 1\}$, both equally likely

- A sequence of agents $(n = 1, 2, \ldots)$ making decisions $x_n \in \{0, 1\}$

- Agent $n$ obtains utility 1 if $x_n = \theta$ and utility 0 otherwise

- Each agent has an iid private binary signals $s_n$, where $s_n = \theta$ with probability $> 1/2$

- Agent $n$ knows his signal $s_n$ and the decisions of previous agents $x_1, x_2, \ldots, x_{n-1}$

- Agent $n$ chooses action 1 if

  $$\mathbb{P} (\theta = 1 | s_n, x_1, x_2, \ldots, x_{n-1}) > \mathbb{P} (\theta = 0 | s_n, x_1, x_2, \ldots, x_{n-1})$$

- If $s_1 = s_2 \neq \theta$, then all agents herd and $x_n \neq \theta$ for all agents,

  $$\lim_{n \to \infty} \mathbb{P}(x_n = \theta) < 1$$
Asymptotic Learning on a Line

- More general model studied by [Smith and Sorensen (00)]
- General signals $s_n$
- Private beliefs bounded if the resulting likelihood ratio is bounded away from 0 and $\infty$
- Private beliefs unbounded otherwise
- On the line there is asymptotic learning, $\lim_{n \to \infty} P(x_n = \theta) = 1$, if private beliefs are unbounded
- No asymptotic learning if private beliefs are bounded
Social Networks

• Previous work considers situations where each individual observes all past actions. Thus no study of network topology.

• In practice, most information obtained from an individual’s social network; friends, neighbors, co-workers...

• How does network structure affect learning?

• How to model learning over networks?
Our Model

- Two possible states of the world $\theta \in \{0, 1\}$, both equally likely
- A sequence of agents $(n = 1, 2, \ldots)$ making decisions $x_n \in \{0, 1\}$. Agent $n$ obtains utility 1 if $x_n = \theta$ and utility 0 otherwise
- Each agent has an iid private signal $s_n$ in $S$. The signal is generated according to distribution $F_\theta$, $F_0$ and $F_1$ absolutely continuous with respect to each other
- $(F_0, F_1)$ is the signal structure
- Agent $n$ has a neighborhood $B(n) \subseteq \{1, 2, \ldots, n - 1\}$ and observes the decisions $x_k$ for all $k \in B(n)$. The set $B(n)$ is private information.
- The neighborhood $B(n)$ is generated according to an arbitrary distribution $Q_n$
- $\{Q_n\}_{n \in \mathbb{N}}$ is the network topology and is common knowledge
- A social network consists of the signal structure and network topology
- **Asymptotic Learning:** Under what conditions does $\lim_{n \to \infty} \mathbb{P}(x_n = \theta) = 1$?
Network Topologies

- \( \{Q_n\}_{n \in \mathbb{N}} \) assigns probability 1 to neighborhood \( \{1, 2..., n - 1\} \) for each \( n \in \mathbb{N} \)—line

- \( \{Q_n\}_{n \in \mathbb{N}} \) assigns probability \( 1/n - 1 \) to each one of the subsets of size 1 of \( \{1, 2..., n - 1\} \) for each \( n \in \mathbb{N} \)—random sampling

- \( \{Q_n\}_{n \in \mathbb{N}} \) assigns probability 1 to neighborhood \( \{n - 1\} \) for each \( n \in \mathbb{N} \)

- \( \{Q_n\}_{n \in \mathbb{N}} \) assigns probability 1 to neighborhoods that are subsets of \( \{1, 2, ..., K\} \) for each \( n \in \mathbb{N} \) for some \( K \in \mathbb{N} \)—example of excessively influential agents
Example Network Topology
Related Literature

- **Bayesian Learning**
  - Banerjee (92), Bikhchandani, Hirshleifer and Welch (92), Smith and Sorensen (00)
  - Banerjee and Fudenberg (04), Smith and Sorensen (98), Gale and Kariv (03), Celen and Kariv (04)

- **Boundedly Rational Learning in Networks**
  - Ellison and Fudenberg (93, 95), Bala and Goyal (98, 01)
  - DeMarzo, Vayanos, Zwiebel (03), Golub and Jackson (07)

- **Decentralized Detection**
  - Cover (69), Papastavrou and Athans (90), Tay, Tsitsiklis and Win (06, 07).
Our Contributions

• We study sequential decision-making and information aggregation in social networks

• We establish decision rules used in perfect Bayesian equilibria

• When the signals lead to unbounded private beliefs:
  – We fully characterize the set of network topologies that lead to learning

• When the signals lead to bounded private beliefs:
  – We show most ‘reasonable’ networks do not lead to learning
  – We show learning is possible with stochastic network topologies
Perfect Bayesian Equilibria

- Agent $n$'s information set is $I_n = \{s_n, B(n), x_k \text{ for all } k \in B(n)\}$
- A strategy for individual $n$ is $\sigma_n : \mathcal{I}_n \rightarrow \{0, 1\}$
- A strategy profile is a sequence of strategies $\sigma = \{\sigma_n\}_{n \in \mathbb{N}}$.
  - A strategy profile $\sigma$ induces a probability measure $\mathbb{P}_\sigma$ over $\{x_n\}_{n \in \mathbb{N}}$.

**Definition:** A strategy profile $\sigma^*$ is a pure-strategy Perfect Bayesian Equilibrium if for each $n \in \mathbb{N}$

$$\sigma^*_n(I_n) \in \arg\max_{y \in \{0, 1\}} \mathbb{P}(y, \sigma^*_{-n})(y = \theta \mid I_n)$$

- A pure strategy PBE exists. Denote the set of PBEs by $\Sigma^*$.

**Definition:** Given a signal structure $(\mathcal{F}_0, \mathcal{F}_1)$ and a network topology $\{Q_n\}_{n \in \mathbb{N}}$, we say that asymptotic learning occurs in equilibrium $\sigma$ if $x_n$ converges to $\theta$ in probability (according to measure $\mathbb{P}_\sigma$), that is,

$$\lim_{n \to \infty} \mathbb{P}_\sigma(x_n = \theta) = 1$$
Equilibrium Decision Rule

Lemma: The decision of agent $n$, $x_n = \sigma(I_n)$, satisfies

$$x_n = \begin{cases} 
1, & \text{if } \mathbb{P}_\sigma(\theta = 1 \mid s_n) + \mathbb{P}_\sigma(\theta = 1 \mid B(n), x_k \text{ for all } k \in B(n)) > 1, \\
0, & \text{if } \mathbb{P}_\sigma(\theta = 1 \mid s_n) + \mathbb{P}_\sigma(\theta = 1 \mid B(n), x_k \text{ for all } k \in B(n)) < 1,
\end{cases}$$

and $x_n \in \{0, 1\}$ otherwise.

- The belief about the state decomposes into two parts:
  - the Private Belief: $\mathbb{P}_\sigma(\theta = 1 \mid s_n)$;
  - the Social Belief: $\mathbb{P}_\sigma(\theta = 1 \mid B(n), x_k \text{ for all } k \in B(n))$. 
**Private Beliefs**

**Lemma:** The private belief of agent $n$ is

$$p_n(s_n) = \mathbb{P}_\sigma(\theta = 1|s_n) = \left(1 + \frac{dF_0(s_n)}{dF_1(s_n)}\right)^{-1}.$$

**Definition:** The signal structure has **bounded private beliefs** if there exists some $0 < m, M < \infty$ such that the Radon-Nikodym derivate $dF_0/dF_1$ satisfies

$$m < \frac{dF_0}{dF_1}(s) < M,$$

for almost all $s \in S$ under measure $(F_0 + F_1)/2$. The signal structure has **unbounded private beliefs** if

$$\inf_{s \in S} \frac{dF_0}{dF_1}(s) = 0 \quad \text{and} \quad \sup_{s \in S} \frac{dF_0}{dF_1}(s) = \infty.$$

- Bounded private beliefs $\iff$ bounded likelihood ratio
- If the private beliefs are unbounded, then there exist some agents with beliefs arbitrarily close to 0 and other agents with beliefs arbitrarily close to 1.
There exist signal structures \((F_0, F_1)\) such that for all equilibria \(\sigma\),

\[
P_\sigma (\theta = 1 | x_1 = \ldots = x_4 = 0, x_5 = \ldots = x_7 = 1) >
\]

\[
P_\sigma (\theta = 1 | x_2 = \ldots = x_4 = 0, x_1 = x_5 = \ldots = x_7 = 1)
\]

Need a strategy of analysis not relying on monotonicity
Properties of Network Topology

**Definition:** A network topology \( \{ Q_n \}_{n \in \mathbb{N}} \) has **expanding observations** if for all \( K \),

\[
\lim_{{n \to \infty}} Q_n \left( \max_{{b \in B(n)}} b < K \right) = 0.
\]

Otherwise, it has **nonexpanding observations**

- Expanding observations do not imply connected graph
- Nonexpanding observations equivalently: There exists some \( K, \epsilon > 0 \) and an infinite subset \( \mathcal{N} \subset \mathbb{N} \) such that

\[
Q_n \left( \max_{{b \in B(n)}} b < K \right) \geq \epsilon \quad \text{for all} \quad n \in \mathcal{N}.
\]

- A finite group of agents is **excessively influential** if there exists an infinite number of agents who, with probability uniformly bounded away from 0, observe only the actions of a subset of this group.
  - For example, a group is excessively influential if it is the source of all information for an infinitely large component of the network
- Nonexpanding observations \( \iff \) exceedingly influential agents
Main Results

**Theorem 1:** Assume that the network topology \( \{Q_n\}_{n \in \mathbb{N}} \) has nonexpanding observations. Then, there exists no equilibrium \( \sigma \in \Sigma^* \) with asymptotic learning.

**Theorem 2:** Assume that the signal structure \((\mathbb{F}_0, \mathbb{F}_1)\) has unbounded private beliefs and the network topology \( \{Q_n\}_{n \in \mathbb{N}} \) has expanding observations. Then, asymptotic learning occurs in every equilibrium \( \sigma \in \Sigma^* \).
Deterministic Topologies

- In a deterministic network, \( \pi \) is an information path of agent \( n \) if for each \( i \), \( \pi_i \in B(\pi_{i+1}) \) and the last element of \( \pi \) is \( n \). The information depth \( L(n) \) is the number of elements in the maximal \( \pi(n) \).

**Corollary:** Assume that the signal structure \((F_0, F_1)\) has unbounded private beliefs and that the network topology is deterministic. Then, asymptotic learning occurs for all equilibria if and only if \( \{L(n)\}_{n \in \mathbb{N}} \) goes to infinity.
Proof Idea of Theorem 1

• Since nonexpanding observations, there exists some $K$, $\epsilon > 0$ and an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that

$$Q_n \left( \max_{b \in B(n)} b < K \right) \geq \epsilon \text{ for all } n \in \mathcal{N}.$$  

• Then, for any $n \in \mathcal{N}$ and any equilibrium $\sigma$,

$$P_\sigma(x_n = \theta) = P_\sigma \left( x_n = \theta \mid \max_{b \in B(n)} b < K \right) Q_n \left( \max_{b \in B(n)} b < K \right)$$

$$+ P_\sigma \left( x_n = \theta \mid \max_{b \in B(n)} b \geq K \right) Q_n \left( \max_{b \in B(n)} b \geq K \right)$$

$$\leq 1 - \epsilon + \epsilon P_\sigma \left( x_n = \theta \mid \max_{b \in B(n)} b < K \right)$$

• Let $f$ give the best estimate of the state given a finite set of iid signals

$$P_\sigma \left( x_n = \theta \mid \max_{b \in B(n)} b < K \right) \leq P( f(s_1, s_2, ..., s_{K-1}, s_n) = \theta ) < 1$$

• The result follows
Proof of Theorem 2: Roadmap

- Characterization of equilibrium strategies when observing a single agent
- **Strong improvement principle** when observing one agent
- **Generalized strong improvement principle**
- Asymptotic learning with unbounded private beliefs and expanding observations
Observing a Single Decision

- Given $\sigma$ and $n$, let us define $Y_n^\sigma$ and $N_n^\sigma$ as
  \[ Y_n^\sigma = \mathbb{P}_\sigma(x_n = 1 \mid \theta = 1), \quad N_n^\sigma = \mathbb{P}_\sigma(x_n = 0 \mid \theta = 0). \]

- The unconditional probability of a correct decision is
  \[ \frac{1}{2}(Y_n^\sigma + N_n^\sigma) = \mathbb{P}_\sigma(x_n = \theta) \]

- We also define the thresholds $L_n^\sigma$ and $U_n^\sigma$ in terms of these probabilities:
  \[ L_n^\sigma = \frac{1 - N_n^\sigma}{1 - N_n^\sigma + Y_n^\sigma}, \quad U_n^\sigma = \frac{N_n^\sigma}{N_n^\sigma + 1 - Y_n^\sigma}. \]

**Proposition:** Let $B(n) = \{b\}$ for agent $n$. Agent $n$'s decision $x_n$ in $\sigma \in \Sigma^*$ satisfies

\[
x_n = \begin{cases} 
0, & \text{if } p_n < L_b^\sigma \\
x_b, & \text{if } p_n \in (L_b^\sigma, U_b^\sigma) \\
1, & \text{if } p_n > U_b^\sigma.
\end{cases}
\]
Observing a Single Decision (continued)

- Let the conditional distribution of private belief $p$ be

$$\mathbb{G}_j(r) = \mathbb{P}(p \leq r \mid \theta = j)$$

- Let $\underline{\beta}$ and $\overline{\beta}$ be the lower and upper support of private beliefs

- Equilibrium decisions:
Strong Improvement Principle

- Agent $n$ has the option of copying the action of any agent in his neighborhood:

$$
P_\sigma(x_n = \theta \mid B(n) = \mathcal{B}) \geq \max_{b \in \mathcal{B}} P_\sigma(x_b = \theta).$$

- Similar to the welfare improvement principle in Banerjee and Fudenberg (04) and Smith and Sorensen (98), and imitation principle in Gale and Kariv (03)

- Using the equilibrium decision rule and the properties of private beliefs, we establish a strict gain of agent $n$ over agent $b$.

**Proposition: (Strong Improvement Principle)** Let $B(n) = \{b\}$ for some $n$ and $\sigma \in \Sigma^*$ be an equilibrium. There exists a continuous, increasing function $Z : [1/2, 1] \rightarrow [1/2, 1]$ with $Z(\alpha) \geq \alpha$ such that

$$P_\sigma(x_n = \theta \mid B(n) = \{b\}) \geq Z(P_\sigma(x_b = \theta)).$$

If the private beliefs are unbounded, then:

- $Z(\alpha) > \alpha$ for all $\alpha < 1$

- $\alpha = 1$ is the unique fixed point of $Z(\alpha)$
Generalized Strong Improvement Principle

- When multiple agents in the neighborhood, learning no worse than observing just one of them:

**Proposition (Generalized Strong Improvement Principle)** For any $n \in \mathbb{N}$, any set $\mathcal{B} \subseteq \{1, ..., n - 1\}$ and any equilibrium $\sigma \in \mathcal{S}$, we have

$$\mathbb{P}_\sigma (x_n = \theta \mid B(n) = \mathcal{B}) \geq \mathcal{Z} \left( \max_{b \in \mathcal{B}} \mathbb{P}_\sigma (x_b = \theta) \right).$$

**Proof of Theorem 2**

- Under expanding observations, one can construct a sequence of agents along which the generalized strong improvement principle applies

- Unbounded private beliefs imply that along this sequence $\mathcal{Z}(\alpha)$ strictly increases

- Until unique fixed point $\alpha = 1$, corresponding to asymptotic learning
No Learning under Bounded Beliefs

**Theorem 3:** If the private beliefs are bounded and the network topology satisfies one of the following conditions,

(a) \( B(n) = \{1, ..., n - 1\} \) for all \( n \),

(b) \( |B(n)| \leq 1 \) for all \( n \),

(c) there exists some constant \( M \) such that \( |B(n)| \leq M \) for all \( n \) and

\[
\lim_{n \to \infty} \max_{b \in B(n)} b = \infty \text{ with probability 1,}
\]

then asymptotic learning does not occur.

- **Implication:** No learning with random sampling and bounded beliefs

**Proof Idea - Theorem 3(c):**

- Asymptotic learning implies social beliefs converge to 0 or 1 almost surely

- But with bounded beliefs, this implies individuals decide on the basis of social belief alone

- Then, positive probability of mistake–contradiction
Learning under Bounded Beliefs

**Theorem 4:** There exist network topologies where asymptotic learning occurs for any signal structure \((\mathcal{F}_0, \mathcal{F}_1)\).

- In the paper, characterization of a class of network topologies for which asymptotic learning occurs with bounded beliefs

**Example:** For all \(n\),

\[
B(n) = \begin{cases} 
\{1, \ldots, n-1\}, & \text{with probability } 1 - r(n); \\
\emptyset, & \text{with probability } r(n),
\end{cases}
\]

for some sequence \(\{r(n)\}\) where \(\lim_{n \to \infty} r(n) = 0\) and \(\sum_{n=1}^{\infty} r(n) = \infty\).

In this case, asymptotic learning occurs for an arbitrary signal structure \((\mathcal{F}_0, \mathcal{F}_1)\) and at any equilibrium.
Proof Idea

• Individuals with empty neighborhood must act according to their private beliefs

• If they are identified by a marker, then simply apply weak law of large numbers

• For the stochastic network topology, we prove that eventually all agents with $B(n) = \{1, \ldots, n - 1\}$ converge on a decision using martingale convergence.

• Eventually, everyone can identify the agents with $B(n) = \emptyset$ and extract true state from them using weak law of large numbers.
Summary

- When does asymptotic learning occur?

<table>
<thead>
<tr>
<th></th>
<th>Unbounded Beliefs</th>
<th>Bounded Beliefs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expanding Observations</td>
<td>YES</td>
<td>USUALLY NO, SOMETIMES YES</td>
</tr>
<tr>
<td>Other Topologies</td>
<td>NO</td>
<td>NO</td>
</tr>
</tbody>
</table>

- No asymptotic learning with unbounded beliefs due to excessively influential agents

- If there is a group of agents who are “influential”, but not excessively so (for example, overrepresented in the information sets of others), this does not prevent asymptotic learning with unbounded beliefs ⇒ contrast with myopic learning
Future Directions

• How does the rate of learning with unbounded beliefs depend on network topology?

• With bounded beliefs, how does the structure of the social network affect probability of wrong asymptotic beliefs?

• Learning in social networks with repeated actions and observations

• How does network structure interact with learning when underlying state is changing?

• Heterogeneous preferences